LECTURE 6

Explicit Solutions of the Heat Equation

Recall the 1-dimensional homogeneous Heat Equation

$$u_t - a^2 u_{xx} = 0 \quad . \tag{1}$$

In this lecture our goal is to construct explicit solutions to (1) satisfying boundary conditions of the form

$$u(x,0) = f(x) \qquad , \qquad -\infty < x < +\infty \tag{2}$$

that will be valid for all $t > 0$. Physically, this problem corresponds to determining the temperature of an infinitely long wire at arbitrary points x and times t, given an initial temperture distribution $f(x)$.

To follow this goal we will follow the usual approach in the theory of differential equations:

- Find (or even guess) some basic solutions.
- Figure out how to construct even more solutions.
- Figure out how to satisfy the boundary condition using your collection of known solutions.

1. Five ways of generating more solutions to the Heat Equation

We will tackle the second step first.

1.1. The Superposition Principle.

Now first of all, since the Heat Equation (1) is a homogeneous linear PDE, we can always create more solutions by taking linear combinations of solutions:

THEOREM 6.1. *If $u_1(x_1,\dots,x_n)$ and $u_2(x_1,\dots,x_n)$ are two solutions of a homogeneous linear differential equation then any function $u(x,t)$ of the form*

$$u(x_1,\dots,x_n) = c_1 u_1(x_1,\dots,x_n) + c_2 u_2(x_1,\dots,x_n) \qquad , \qquad c_1, c_2 \ \text{constants} \quad ,$$

will also be a solution.

While this theorem is extremely useful in developing the general solution of linear ODEs (i.e. when $n = 1$), it's a bit weak for developing the general solution of second order linear PDEs. For example, in the case of a second order linear ODE, the entire solution space space is 2-dimensional and so if $u_1(x)$ and $u_2(x)$ are linearly independent solutions, **all** the solutions can be constructed from linear combinations of $u_1(x)$ and $u_2(x)$. The solution space of a linear PDE, however, is infinite-dimensional (the general solution of a PDE will in general involve arbitrary functions rather than arbitrary constants) and so there is no hope of getting all the solutions by forming linear combinations of two or even a finite number of solutions.

The constructions I give below will be much more useful – each will provide independent solutions parameterized by a continuous variable; allowing us to generate infinitely many independent solutions from a given solution.

1.2. Generating solutions by translation.

LEMMA 6.2. *Suppose $u(x,t)$ is a solution of the homogeneous heat equation (1). Then for any fixed constant $y \in \mathbb{R}$,*

$$\phi := u(x-y,t)$$

is also a solution of (1).

Proof. Let $u(x,t)$ be a solution of (1) and set $\phi(x,y) = u(x-y,t)$ for some $y \in \mathbb{R}$. Now by the Chain Rule

$$\frac{\partial}{\partial x} u(z(x),t) = \left.\frac{\partial u}{\partial z}\right|_{z=z(x)} \frac{dz}{\partial x} + \left.\frac{du}{\partial t}\right|_{z=z(x)} \frac{dt}{dx} = \frac{dz}{dx}\left(\left.\frac{\partial u}{\partial z}\right|_{z=z(x)}\right)$$

so when $z(x) = y - x$, we have

$$\frac{\partial}{\partial x} u(x-y,t) = \left(\frac{d}{dx}(x-y)\right)\left(\left.\frac{\partial u}{\partial z}\right|_{z=z(x)}\right) = \left.\frac{\partial u}{\partial z}\right|_{z=x-y}$$

Differentiating again with respect to x we similarly get

$$\frac{\partial^2}{\partial x^2} u(x-y,t) = \left.\frac{\partial^2 u}{\partial z^2}\right|_{z=x-y}$$

Let's now plug $\phi(x,t)$ into the left hand side of the Heat Equation:

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(x,t) - a^2 \frac{\partial^2}{\partial x^2}\phi(x,t) &= \frac{\partial}{\partial t} u(x-y,t) - a^2 \frac{\partial^2}{\partial x^2} u(x-y,t) \\
&= u_t(x-y,t) - a^2 \left. u_{xx}(z,t)\right|_{z=x-y} \\
&= u_t(x-y,t) - a_2 u_{xx}(x-y,t)
\end{aligned}$$

Since (1) is to hold at all points $x \in \mathbb{R}$, it hold in particular at the point $x - y \in \mathbb{R}$. Thus,

$$\frac{\partial}{\partial t}\phi(x,t) - a^2 \frac{\partial^2}{\partial x^2}\phi(x,t) = u_t(x-y,t) - a_2 u_{xx}(x-y,t) = 0$$

and so $\phi(x,t)$ also satisfies the Heat Equation. □

1.3. Generating solutions by taking partial derivatives.

LEMMA 6.3. *Suppose $u(x,t)$ is a solution of the homogeneous Heat Equation (1). Then any partial derivative of $u(x,t)$ is also a solution.*

Proof. Let $u(x,t)$ be a solution of (1) and set $v(x,t) = \frac{\partial}{\partial x} u(x,t)$. Then

$$\begin{aligned}
\frac{\partial v}{\partial t} - a^2 \frac{\partial v}{\partial x^2} &= \frac{\partial^2 u}{\partial x \partial t} - a^2 \frac{\partial^3 u}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - a^2 \frac{\partial u}{\partial x^2}\right) \\
&= \frac{\partial}{\partial x}(0) \\
&= 0
\end{aligned}$$

So $v(x,t)$ also satisfies (1). Similarly, if $w(x,t) = \frac{\partial}{\partial t} u(x,t)$, we have

$$\frac{\partial w}{\partial t} - a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial t}\left(u_t - a^2 u_{xx}\right) = \frac{\partial}{\partial t}(0) = 0$$

Finally, via induction, we can conclude if $\phi(x,t) = \frac{\partial^k}{\partial x^k}\frac{\partial^\ell}{\partial t^\ell} u(x,t)$, then $\phi(x,t)$ also satisfies (1). □

1.4. Generating solutions by taking convolutions.

LEMMA 6.4. *Suppose $u(x,t)$ is a solution of the homogeneous Heat Equation (1) and $g(x)$ is any differentiable function such that*

$$\int_{-\infty}^{+\infty} g(x)\, u(x,t)\, dx$$

converges. Then

$$\phi(x,t) := \int_{-\infty}^{\infty} g(y)\, u(x-y,t)\, dy \tag{3}$$

is also a solution of (1).

Proof.

Let $\phi(x,t)$ be the convolution of a solution $u(x,t)$ by the function $g(x)$ (as in (3) above). Since the integral (3) converges we have, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(x,t) - a^2\frac{\partial}{\partial x^2}\phi(x,t) &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} g(y)\, u(x-y,t)\, dy\right) - a^2\frac{\partial}{\partial x^2}\left(\int_{-\infty}^{\infty} g(y)\, u(x-y,t)\, dy\right) \\
&= \int_{-\infty}^{\infty} g(y)\frac{\partial}{\partial t}u(x-y,t)\, dy - a^2\int_{-\infty}^{\infty} g(y)\frac{\partial^2}{\partial x^2}u(x-y,t)\, dy \\
&= \int_{-\infty}^{\infty} g(y)\left(u_t(x-y,t) - u_{xx}(x-y,t)\right) dy \\
&= \int_{-\infty}^{\infty} g(y)\,(0)\, dy \\
&= 0
\end{aligned}$$

where we have used Lemma 6.3 to conclude that if $u(x,t)$ is a solution of the Heat Equation, so is $u(x-y,t)$. Thus, $\phi(x,t)$ also satisfies (1) and we are done. ☐

REMARK 6.5. Although this method of generating solutions by taking convolutions may appear as strangest yet, it can be thought of as a combination of the Superposition Principle and the translation method. Indeed, Lemmas 6.1 and 6.2, if $u(x,t)$ is a solution and $g(x)$ is a function of x, a sum of the form

$$\phi(x,t) := \sum_{i}^{n} g(y_i)\, u(x-y_i,t)\,\Delta y$$

will also be a solution. Interpreting the right hand side as a Riemann sum and taking the limit as n goes to ∞ we recover (3).

1.5. Generating solutions by dilations.

LEMMA 6.6. *Suppose $u(x,t)$ is a solution of the homogeneous Heat Equation (1) and λ is a positive real number. Then the* ***dilated*** *function*

$$v(x,t) := u\left(\lambda x, \lambda^2 t\right)$$

is also a solution of (1).

Proof. We have (more or less the same as the Chain Rule calculation in Lemma 6.2, but using more shorthand notation)

$$\begin{aligned}
\frac{\partial}{\partial t}v(x,t) &= \frac{\partial}{\partial t}u\left(\lambda x,\lambda^2 t\right) = \frac{d\left(\lambda^2 t\right)}{dt}\left(\frac{\partial u}{\partial t}\right) = \lambda^2 u_t\left(\lambda x,\lambda^2 t\right) \\
\frac{\partial^2}{\partial x^2}v(x,t) &= \frac{\partial}{\partial x}\left(\lambda\frac{\partial u}{\partial x}\left(\lambda x,\lambda^2 t\right)\right) = a^2\frac{\partial^2}{\partial x^2}u\left(\lambda x,\lambda^2 t\right)
\end{aligned}$$

and so

$$v_t(x,t) - a^2 v_{xx}(x,t) = \lambda^2 u_t\left(\lambda x, \lambda^2 t\right) - \lambda^4 u_{xx}\left(\lambda x, \lambda^2 t\right) = \lambda^2\left(u_t\left(\lambda x, \lambda^2 t\right) - u_{xx}\left(\lambda x, \lambda^2 t\right)\right) = \lambda^2(0) = 0$$

and so $v(x,t) = u\left(\lambda x, \lambda^2 t\right)$ is also a solution of the Heat Equation (1). □

2. Finding a fundamental solution of the Heat Equation

We'll now turn the first step of our program for solving general Heat Equation problems: finding a basic solution from which we can build lots of other solutions.

Recall the trick that we used to solve a first order linear PDEs

$$A(x,y)\,\phi_x + B(x,y)\,\phi_y = C(x,y)\,\phi$$

was to interprete the right hand side as telling us how solution varied along particular curves. This intepretation allowed us to solve the PDE by solving the ODE characterizing the behavior of the solution along these curves.

Unfortunately, this geometric idea won't help us in the second order situation. However, some semblance of this method will. We will again try to construct a solution of PDE by solving an ODE. This time, however, our rationale for reducing to an ODE will be different.

The basic idea we'll use here is *symmetry*. Forget about the Heat Equation for the moment and suppose you had a PDE in two variables x, y that had solutions that were invariant under rotations $x \to x\cos\theta - y\sin\theta$, $y \to x\sin\theta + y\cos\theta$. Then such a solution would not depend on θ and by converting to polar coordiantes r, θ you should be able to eliminate the angular variable and reduce your PDE an ODE with respect to the radial variable r.

Returning to the Heat Equation, we cannot expect solutions that are rotationally invariant (as there is no natural way to rotate in the x, t plane when x is a spatial coordinate and t is a temporal coordinate). However, we can apply the dilatation operation to solutions of the Heat Equation and remain in the solution space. In fact, we may even be able to find a solution of the Heat Equation such that

$$u\left(ax, a^2 t\right) = u(x,t)$$

In what follows below, we will suppose a functional form for $u(x,t)$ that guarantees this dilatation propery and then require that it also satisfies the Heat Equation. This latter requirement will reduce to a simple ODE that we can solve and then use to construct our basic solution of the Heat Equation.

With a prescient eye towards a normalization that will prove convenient later on, suppose

$$u(x,t) = g\left(\frac{x}{2a\sqrt{t}}\right) \tag{4}$$

any function of this form will be automatically invariant under the dilatation operation

$$u\left(\lambda x, \lambda^2 t\right) = g\left(\frac{(\lambda x)}{2a\sqrt{\lambda^2 t}}\right) = g\left(\frac{x}{2a\sqrt{t}}\right) = u(x,t)$$

Let's see if we can find a function $g(p)$ so that $u(x,t)$ so defined also satisfies the Heat Equation. Thus, we let

$$\begin{aligned} p &= \frac{x}{2a\sqrt{t}} \\ u(x,t) &= g(p(x,t)) \end{aligned}$$

and demand

$$u_t - a^2 u_{xx} = 0 \quad .$$

We have

$$u_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = \frac{dg}{dp}\left(-\frac{1}{2}\frac{x}{2a}t^{-3/2}\right) = -\frac{1}{2t}\frac{x}{2a\sqrt{t}}g'(p) = -\frac{1}{2t}pg'(p)$$

$$u_x = \frac{dg}{dp}\frac{dp}{dx} = \frac{1}{2a\sqrt{t}}g'(p)$$

$$u_{xx} = \frac{1}{4a^2t}g''(p)$$

and so, our demand amounts do

$$0 = u_t - a^2u_{xx} = -\frac{1}{2t}pg(p) - a^2\left(\frac{1}{4a^2t}g''(p)\right) = -\frac{1}{4t}\left(g''(p) + 2g'(p)\right)$$

This demand can be met by finding a function $g(p)$ satisfying

$$g'' + 2pg' = 0 \tag{5}$$

Although this equation is ostensibly 2^{nd} order it can be reduced to solving (and then integrating) a first order ODE. Let $h(p) = g'(p)$. Then $h(p)$ should satisfy

$$h' + 2ph = 0 \quad .$$

This is a first order linear ODE whose general solution is given by

$$h(p) = c_1 \exp\left(-2\int pdp\right) = c_1 \exp\left(-p^2\right) = c_1 e^{-p^2}$$

Now we set

$$g(p) = \int \frac{dg}{dp}dp + c_2 = \int h(p)\,dp + c_2 = c_1\int e^{-p^2}dp + c_2$$

and we'll have the general solution of (5).

Recall that the differential equation for $g(p)$ was ensure that $u(x,t) = g\left(\frac{x}{2a\sqrt{t}}\right)$ would satisfy the Heat Equation. And so, we have

$$u(x,t) = g\left(\frac{x}{2a\sqrt{t}}\right) = c_1\int^{p=\frac{x}{2a\sqrt{t}}} e^{-p^2}dp + c_2 \tag{6}$$

as one solution of the Heat Equation.

The solution (6) involves two arbitrary constants. As in the remark following Lemma 6.1, we can not expect to get a general solution by simply varying the constants c_1 and c_2. In fact, we will now fix them just so that our other methods of generating solutions will be easier notation-wise. The following choices will do that for us

$$c_1 = \frac{1}{\sqrt{\pi}}$$
$$c_2 = \frac{1}{2}$$

Lemma 6.7. *Let*

$$Q(x,t) = \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-p^2}dp + \frac{1}{2}$$

Then, $Q(x,t)$ is the unique solution of the Heat Equation (1) satsifying the boundary condition

$$\lim_{t\to 0^+} u(x,t) = \begin{cases} 0 & if \quad x<0 \\ 1 & if \quad x>0 \end{cases}$$

By construction $Q(x,t)$ is a solution of the Heat Equation (since it is of the form (6)). We have

$$\begin{aligned}\lim_{t\to 0^+} Q(x,t) &= \frac{1}{\sqrt{\pi}}\lim_{t\to 0^+}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-p^2}dp + \frac{1}{2} \\ &= \frac{1}{\sqrt{\pi}}\lim_{L\to+\infty}\int_0^{Lsign(x)} e^{-p^2}dp\end{aligned}$$

or

$$\lim_{t\to 0^+} Q(x,t) == \begin{cases} \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-p^2}dp + \frac{1}{2} & \text{if} \quad x>0 \\ \frac{1}{\sqrt{\pi}}\int_0^{-\infty} e^{-p^2}dp + \frac{1}{2} & \text{if} \quad x<0 \end{cases} \tag{7}$$

The integral

$$\int_0^{\infty} e^{-p^2}dp$$

is a particularly famous one: it is called either the Gaussian integral or the Euler-Poisson integral. It is integrated by viewing its value as the part of an easier integral in two dimensions. Consider

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \int_{-\infty}^{-\infty}\int_{-\infty}^{+\infty} e^{-x^2}e^{-y^2}dxdy = \int_{-\infty}^{+\infty} e^{-x^2}dx\int_{-\infty}^{+\infty} e^{-y^2}dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$$

(in the penultimate step we are merely realizing that the integral over y is identical to the integral over x and in the last step we are using the fact the e^{-x^2} is an even function of x). On the other hand, if we convert to polar coordinates we can can actually carry out the integration over $\mathbb{R}^2$

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy &= \int_0^{+\infty}\int_0^{2\pi} e^{-r^2}rdrd\theta = \\ &= 2\pi\int_{-}^{\infty} re^{-r^2}dr \\ &= \pi\int_0^{\infty} e^{-u}du \qquad (u=r^2) \\ &= -\pi e^{-u}\big|_0^{\infty} \\ &= 0+\pi \\ &= \pi\end{aligned}$$

Thus we have

$$\pi = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2$$

or

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi} \quad .$$

Finally, since e^{-x^2} is an even function of x we have

$$\int_0^{+\infty} e^{-x^2}dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x^2}dx = \frac{\pi}{2}$$

We also need a value for

$$\int_0^{-\infty} e^{-x^2}dx$$

This can be had by making a change of variable $x\to -x$.

$$\int_0^{-\infty} e^{-x^2}dx = -\int_0^{+\infty} e^{-x^2}dx = -\frac{\sqrt{\pi}}{2}$$

Returning to equation (7)

$$\lim_{t\to 0^+} Q(x,t) = \begin{cases} \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-p^2}dp + \frac{1}{2} & \text{if} \quad x>0 \\ \frac{1}{\sqrt{\pi}}\int_0^{-\infty} e^{-p^2}dp + \frac{1}{2} & \text{if} \quad x<0 \end{cases} = \begin{cases} \frac{1}{2}+\frac{1}{2} & \text{if} \quad x>0 \\ \frac{1}{2}-\frac{1}{2} & \text{if} \quad x<0 \end{cases} = \begin{cases} 1 & \text{if} \quad x>0 \\ 0 & \text{if} \quad x<0 \end{cases}$$

which is the desired result. □

3. Solution to the general boundary value problem

We now have a solution

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-p^2} dp + \frac{1}{2}$$

of the Heat Equation that satisfies

$$u(x,0) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

What we really want, however, is a solution satisfying the more general boundary condition

$$u(x,0) = f(x) \tag{8}$$

To find such a solution, we employ the methods of §2 to manipulate our fundamental $Q(x,t)$ until it satisfies (8).

Using Lemma 6.3, the function

$$\begin{aligned} S(x,t) &= \frac{\partial}{\partial x} Q(x,t) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-p^2} dp + \frac{1}{2}\right) \\ &= \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-p^2} dp \end{aligned}$$

The partial derivative is evaluated using the Chain Rule for a function of the single variable and the Fundamental Theorem of Calculus. Upon computation one gets

$$S(x,t) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2a\sqrt{t}}\right) e^{-\left(\frac{x}{2a\sqrt{t}}\right)^1} = \frac{1}{\sqrt{4\pi a^2 t}} e^{-\frac{x^2}{4a^2 t}}$$

Next, we'll use Lemma 6.3 to produce yet another solution

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t) f(y)\, dy \tag{9}$$

This is going to be the desired solution. It just remains to demonstrate that it satisfies the boundary condiations (7).

We have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} S(x-y,t) f(y)\, dy \\ &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} Q(x-y,t) f(y)\, dy \\ &= -\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} Q(x-y,t) f(y)\, dy \\ &= -Q(x-y) f(y)\big|_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} Q(x-y,t) f'(y)\, dy \end{aligned}$$

where we employed integration by parts in the last step. Assuming

$$\lim_{y\to\pm\infty} f(y) = 0$$

(equivalent to requiring that the heat energy stored in the wire at time $t = 0$ is finite), we have

$$u(x,t) = \int_{-\infty}^{+\infty} Q(x-y,t)\, f'(y)\, dy$$

In the limit $t \to 0^+$, we have

$$\lim_{t\to 0^+} u(x,t) = \int_{-\infty}^{+\infty} \lim_{t\to 0^+} Q(x-y,t)\, f'(y)\, dy$$

Recall that $Q(x,t)$ obeys

$$\lim_{t\to 0^+} Q(x,t) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Thus,

$$\begin{aligned}
\lim_{t\to 0^+} u(x,t) &= \int_{-\infty}^{x} Q(x-y,0)\, f'(y)\, dy + \int_{x}^{\infty} Q(x-y,0)\, f'(y)\, dy \\
&= \int_{-\infty}^{x} f'(y)\, dy + 0 \\
&= f(y)|_{-\infty}^{x} \\
&= f(x) - 0 \\
&= f(x)
\end{aligned}$$

We have thus proved

THEOREM 6.8. *The solution to*

$$\begin{aligned}
u_{tt} - a^2 u_{xx} &= 0 \\
u(x,0) &= f(x)
\end{aligned}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi a^2 t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4a^2 t}} f(y)\, dy$$