LECTURE 4

Second Order Linear Equations

1. The Basic Types of 2^{nd} Order Linear PDEs:

1.1. Generic and Standard Forms of 2^{nd} Order Linear PDEs. The generic form of a second order linear PDE in two variables is

$$A(x,y)\frac{\partial^2\phi}{\partial x^2}+B(x,y)\frac{\partial^2\phi}{\partial x\partial y}+C(x,y)\frac{\partial^2\phi}{\partial y^2}+D(x,y)\frac{\partial\phi}{\partial x}+E(x,y)\frac{\partial\phi}{\partial y}+F(x,y)\phi=G(x,y) \tag{1}$$

We shall see in a second that by a suitable change of coordinates $x,y\rightarrow\xi(x,y),\eta(x,y)$ we can cast any PDE of the form (1) into one of the following three (standard) forms.

(P) **Parabolic Equations:**

$$\frac{\partial^2\Phi}{\partial\xi^2}+f_1(\xi,\eta)\frac{\partial\Phi}{\partial\xi}+f_2(\xi,\eta)\frac{\partial\Phi}{\partial\eta}+f_3(\xi,\eta)\Phi=g(\xi,\eta) \tag{2}$$

(E) **Elliptic Equations:**

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}+f_1(\xi,\eta)\frac{\partial\Phi}{\partial\xi}+f_2(\xi,\eta)\frac{\partial\Phi}{\partial\eta}+f_3(\xi,\eta)\Phi=g(\xi,\eta) \tag{3}$$

(H) **Hyperbolic Equations:**

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta}+f_1(\xi,\eta)\frac{\partial\Phi}{\partial\xi}+f_2(\xi,\eta)\frac{\partial\Phi}{\partial\eta}+f_3(\xi,\eta)\Phi=g(\xi,\eta) \tag{4}$$

To show this, it is helpful to rewrite things in a matrix notation. First let's introduce an index notation for our coordinate systems by letting $x\rightarrow x_1$, $y\rightarrow y_1$, $\xi\rightarrow\xi_1$, $\eta\rightarrow\xi_2$ and letting

$$\begin{aligned}A_{11}(\mathbf{x}) &= A_{11}(x_1,x_2)=A(x,y)\\ A_{12}(\mathbf{x}) &= A_{12}(\mathbf{x})=\frac{1}{2}B(x,y)\\ A_{22}(\mathbf{x}) &= C(x,y)\end{aligned}$$

so that the leading terms (the terms with the second order derivatives) of our PDE correspond to the differential operator.

$$\mathcal{L}_2=\sum_{i,j}A_{i,j}(\mathbf{x})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$$

We would like to find a change of variables that simplifies this differential operator as much as possible. Now let us introduce a change of coordinates where the new coordinates $\boldsymbol{\xi}=[\xi_1,\xi_2]$ are expressed as certain functions of the old coordinates

$$\begin{aligned}\xi_2 &= \xi_1(x_1,x_2)\\ \xi_2 &= \xi_2(x_1,x_2)\end{aligned}$$

Now the Chain Rule gives us a rule for constructing the differential operator $\widetilde{\mathcal{L}}_2$ with respect to the new variables that corresponds to the action of the original differential operator $\mathcal{L}_2$. Indeed, the Chain Rule says that

$$\frac{\partial}{\partial x_i} = \sum_a \frac{\partial \xi_a}{\partial x_i} \frac{\partial}{\partial \xi_a}$$

and so

$$\begin{aligned}
\mathcal{L}_2 &= \sum_{i,j} A_{i,j}(\mathbf{x}) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{i,j} A_{ij}(\mathbf{x}) \sum_{a,b} \frac{\partial \xi_a}{\partial x_i} \frac{\partial}{\partial \xi_a} \frac{\partial \xi_b}{\partial x_j} \frac{\partial}{\partial \xi_b} \\
&= \sum_{a,b} \sum_{i,j} \frac{\partial \xi_a}{\partial x_i} A_{ij}(\mathbf{x}) \frac{\partial \xi_b}{\partial x_j} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b} \quad + \text{lower order terms}
\end{aligned}$$

So, in terms of the new variable the second order terms of the differential equation correspond to the differential operator

$$\widetilde{\mathcal{L}}_2 = \sum_{a,b} \sum_{i,j} \frac{\partial \xi_a}{\partial x_i} A_{ij}(\mathbf{x}) \frac{\partial \xi_b}{\partial x_j} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$$

Let us again introduce some more matrix notation by defining a 2×2 matrix $\mathbf{J}$ as

$$\mathbf{J}_{ia} = \left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} \right)_{i,a} = \frac{\partial \xi_a}{\partial x_i} \quad , \quad i, a = 1, 2$$

(We note that the absolute value of the determinant of this matrix is the Jacobian of the coordinate transformation). This then allows us to write $\widetilde{\mathcal{L}}_2$ as

$$\widetilde{\mathcal{L}}_2 = \sum_{a,b} (\mathbf{JAJ})_{ab} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$$

In short, after making a change of coordinates the coefficient matrix of the second order terms is related to the original coefficient matrix $\mathbf{A}$ by

$$\mathbf{A} \to \mathbf{JAJ}$$

where

$$\mathbf{J}_{ia} \equiv \frac{\partial \xi_a}{\partial x_i}$$

We now quote a theorem from linear algebra:

THEOREM 4.1. *If* $\mathbf{A}$ *is a real symmetric matrix, there is a orthogonal matrix* $\mathbf{O}$ *such that* $\mathbf{OAO}$ *is a diagonal matrix.*

Thus, by now choosing our coordinate transformation so that the matrix $\mathbf{J}$ corresponds to a suitable orthogonal matrix, we can send the matrix $\mathbf{A}$ to a diagonal matrix:

$$\mathbf{A} \to \mathbf{JAJ} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

In fact, with a little more work, one can find coordinate transformations so that $\mathbf{JAJ}$ takes one of the following three forms

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad .$$

To understand, why we should have at least these three case, we note that the if $\mathbf{J}$ is an orthogonal matrix $\mathbf{A}$ has the same determinant as $\mathbf{JAJ}$. The three matrices above have determinants of, respectively, 1, -1 and 0. These three cases correspond to the situations where the original matrix $\mathbf{A}$ has a positive, negative or zero determinant.

So let me state cleanly the general situation for second order linear PDEs.

PROPOSITION 4.2. *If*

$$\sum_{i,j} A(\mathbf{x})_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_i B(\mathbf{x})_i \frac{\partial \phi}{\partial x_i} + C(\mathbf{x})\phi = D(\mathbf{x})$$

is a second order linear PDE, there are three possibilities depending on the sign of the determinant of the matrix $\mathbf{A}(x)$:

Elliptic Case : *If* $\det \mathbf{A}(x) > 0$*, there is a coordinate transformation* $\mathbf{x} \to \boldsymbol{\xi}$ *that sends the PDE to a PDE of the form*

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + f_1(\xi,\eta)\frac{\partial \Phi}{\partial \xi} + f_2(\xi,\eta)\frac{\partial \Phi}{\partial \eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta)$$

Hyperbolic Case : *If* $\det \mathbf{A}(x) < 0$*, there is a coordinate transformation* $\mathbf{x} \to \boldsymbol{\xi}$ *that sends the PDE to a PDE of the form*

$$\frac{\partial^2 \Phi}{\partial \xi^2} - \frac{\partial^2 \Phi}{\partial \eta^2} + f_1(\xi,\eta)\frac{\partial \Phi}{\partial \xi} + f_2(\xi,\eta)\frac{\partial \Phi}{\partial \eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta)$$

Parabolic Case : *If* $\det \mathbf{A}(x) > 0$*, there is a coordinate transformation* $\mathbf{x} \to \boldsymbol{\xi}$ *that sends the PDE to a PDE of the form*

$$\frac{\partial^2 \Phi}{\partial \xi^2} + f_1(\xi,\eta)\frac{\partial \Phi}{\partial \xi} + f_2(\xi,\eta)\frac{\partial \Phi}{\partial \eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta)$$

2. The Basic Prototypes

Associated to each of these standard forms are prototypical examples, each of which also corresponds to a fundamental PDE occuring in physical applications. For the next few weeks we shall discuss the solutions or each of these equations extensively.

2.1. The Heat Equation.

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{5}$$

This equation arises in studies of heat flow. For example, if a 1-dimensional wire is heated at one end, then the function $\phi(x,t)$ describing the temperature of the wire at position x and time t will satisfy (5). The heat equation is the prototypical example of a parabolic PDE.

2.2. Laplace's Equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{6}$$

This equation arises in a variety of physical situations: the function $\phi(x,y)$ might be interpretable as the electric potential at a point (x,y) in the plane, or the steady state temperature of a point in the plane. Laplace's equation is the prototypical example of an elliptic PDE.

2.3. The Wave Equation.

$$\frac{\partial^2 \phi}{\partial t^2} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{7}$$

This equation governs the propagation of waves in a medium, such as the vibrations of a taunt string, pressure fluctuations in a compressible fluid, or electromagnetic waves. The wave equation is the prototypical example of a hyperbolic PDE. The coordinate transformation that casts (7) into the form (4) is

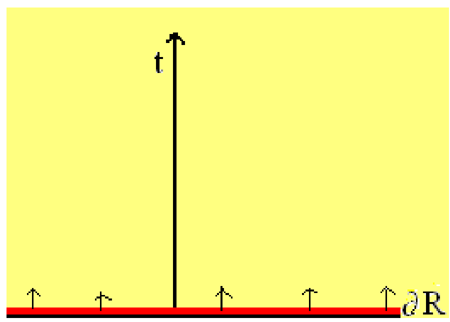
$$\begin{aligned} \xi &= x - ct \\ \eta &= x + ct \end{aligned}$$

3. Boundary Conditions

In stark constrast to the theory of ordinary differential equations where boundary conditions play a relatively innocuous role in the construction of solutions, the nature of the boundary conditions imposed on a partial differential equation can have a dramatic effect on whether or not the PDE/BVP (partial differential equation / boundary value problem) is solvable.

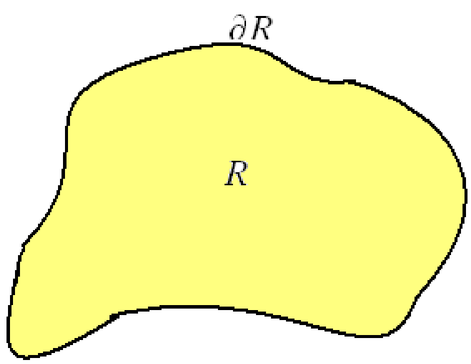
There three particular kinds of boundary conditions that are particularly common in physical applications.

3.1. Cauchy Conditions. Suppose $\mathcal{L}[\phi] = G(\mathbf{x}, \phi)$ is a PDE imposed on a region $R \subset \mathbb{R}^n$ with boundary ∂R.. Cauchy boundary conditions in such a situation would be the specification of the function and its normal derivative along the boundary curve.



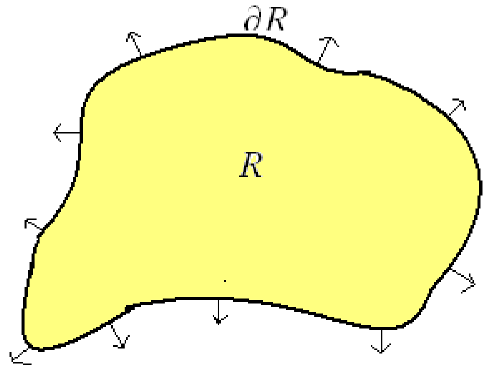
Cauchy boundary conditions are commonly applicable in dynamical situations (where the system is interpreted as evolving with respect to a time parameter t.

3.2. Dirichlet Conditions. The specification of the function on the boundary curve.



As an example of a PDE/BVP with Dirichlet boundary conditions, consider the problem of finding the equilibrium temperature distribution of a rectangular sheet whose edges are maintained at some prescribed (but non-constant) temperature.

3.3. Neumann Conditions. The specification of the normal derivative of the function along the boundary curve.



As an example of a PDE/BVP with Neumann boundary conditions, consider the problem of determining the electric potential inside a superconducting cylinder.