LECTURE 3

Characteristics and First Order Equations

We shall now generalize the methods developed in the preceding lecture.

DEFINITION 3.1. *A partial differential equation in n variables, x_i is said to be* ***quasi-linear*** *if it is linear in the partial derivatives of the unknown function. Thus, a quasi-linear PDE is an equation of the form*

$$\sum_{i=1}^{n} F_i u_{x_i} = G \tag{1}$$

where the coefficients F_i and G are given functions of the coordinates x_i and the unknown function u. Note that we do not require the F_i or G to be linear in u.

The case we shall focus on is the case where there only two underlying variables x and y; in which case we shall write

$$A(x,y,u)\,u_x + B(x,y,u)\,u_y = C(x,y,u)$$

as the generic form of a quasi-linear first order PDE.

We shall associate with such a PDE two geometric objects leaving in a space of dimension $n+1$ ($=3$ when we have only two underlying variables). The first are the **solution surfaces**. If $\phi(x_1,\ldots,x_n)$ is a solution of (1), then the solution surface for ϕ is

$$S_\phi = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_{n+1} = \phi(x_1,\ldots,x_n) \right\}.$$

This coincides with the graph of $\phi(x_1,\ldots,x_n)$ as a function of n variables.

The second geometric object is actually a family of curves $\boldsymbol{\gamma}(t) : \mathbb{R} \to \mathbb{R}^n$. Writing $\boldsymbol{\gamma}(t) = [x_1(t), x_2(t), \cdots, x_n(t), u(t)]$, we require the component functions $x_i(t)$ and $u(t)$ to satisfy the following system of differential equations

$$\begin{aligned}
\frac{dx_1}{dt}(t) &= F_1(x_1(t),\ldots,x_n(t),u(t)) \quad , \\
\frac{dx_2}{dt}(t) &= F_2(x_1(t),\ldots,x_n(t),u(t)) \quad , \\
&\vdots \\
\frac{dx_n}{dt}(t) &= F_n(x_1(t),\ldots,x_n(t),u(t)) \quad , \\
\frac{du}{dt}(t) &= G(x_1(t),\ldots,x_n(t),u(t)) \quad ,
\end{aligned} \tag{2}$$

where the functions $F_1, F_2, \ldots, F_n, G$ appearing on the right hand sides are the same functions that appear in the quasi-linear PDE (1). Such a system of ordinary differentials equations is called an *autonomous system*; this just means a system of ODEs where the underlying variable t never appears explicitly in the differential equations.

DEFINITION 3.2. *A* ***characteristic curve*** *for a quasi-linear PDE*

$$\sum_{i=1}^{n} F_i u_{x_i} = G$$

is a curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^{n+1}$ satisfying the equations (2) above.

Let me now recall the existence and uniqueness theorem for systems of first order ODEs. But before doing so, let's first simplify our setting to case where we have a quasilinear PDE with only two underlying variables

$$A(x,y,u)\,u_x + B(x,y,u)\,u_y = G(x,y,y)$$

(The general case can be treated in essentially the same way, it's just that the notation is more cumbersome.)

Theorem 3.3 (Existence and Uniqueness for Systems of Autonomous ODEs). *Let Ω be a region in $\mathbb{R}^3$ for which the functions $A(x,y,u)$, $B(x,y,u)$ and $G(x,y,u)$ are continuous and differentiable. Then through each point $(x_o, y_o, u_o) \in \Omega$, there is precisely one solution of*

$$\begin{aligned} \frac{dx}{dt}(t) &= A(x(t), y(t), u(t)) \\ \frac{dy}{dt}(t) &= B(x(t), y(t), u(t)) \\ \frac{du}{dt}(t) &= G(x(t), y(t), u(t)) \end{aligned} \tag{3}$$

satisfying

$$\begin{aligned} x(0) &= x_o \\ y(0) &= y_o \\ u(0) &= u_o \end{aligned} \tag{4}$$

Thus, one we fix a quasi-linear PDE (and so an autonomous system of ODEs) and a point (x_o, y_o, u_o) we are guaranteed a unique characteristic curve $\boldsymbol{\gamma}(t)$ that passes through $[x_o, y_o, u_o]$ at $t = 0$.

Here is the fundamental fact about characteristics and solution surfaces.

Proposition 3.4. *Suppose ϕ is a solution of a quasi-linear PDE*

$$\sum_{i=1}^{n} F_i \phi_{x_i} = G \quad .$$

Let $S = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_{n+1} = \phi(x_1, \cdots, x_n) \right\}$ be corresponding surface in $\mathbb{R}^{n+1}$. Then if a characteristic curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ passes through one point of S, it lies entirely in S.

Proof. From Vector Calculus we know that the vector

$$\mathbf{n}(x_1, \cdots, x_n) = \left(\phi_{x_1}(x_1, \cdots, x_n), \cdots, \phi_{x_n}(x_1, \cdots, x_n), -1 \right) \in \mathbb{R}^{n+1}$$

represents the direction (in $\mathbb{R}^{n+1}$) of the normal to the surface S above the point $(x_1, \cdots, x_n) \in \mathbb{R}^n$. We also know that the tangent vector (in $\mathbb{R}^{n+1}$) to a characteristic $\boldsymbol{\gamma}(t)$ is given by

$$\frac{d\boldsymbol{\gamma}}{dt}(t) = \left(\frac{dx_1}{dt}, \cdots, \frac{dx_n}{dt}, \frac{du}{dt} \right) = \left(F_1(\boldsymbol{\gamma}(t)), \cdots, F_n(\boldsymbol{\gamma}(t)), G(\boldsymbol{\gamma}(t)) \right) \quad .$$

But then if $\gamma(t) = (x_1, \cdots, x_n, \phi(x_1, \cdots, x_n))$ we have

$$\frac{d\boldsymbol{\gamma}}{dt}(t) \cdot \mathbf{n}(x_1, \cdots, x_n) = F_1 \phi_{x_1} + \cdots F_n \phi_{x_n} - G = 0$$

by virtue of the original PDE. Thus, the tangent vector to any characteristic passing through a given point on a solution surface always lies in (the tangent plane to) the surface. Thus, if a characteristic passes through a solution surface it can never leave that surface. □

Let me now explain how these results allow us to construct solutions on first order quasi-linear PDEs. To keep our notation simple, we'll restrict to the case of where $n = 2$ and consider the case of a generic quasi-linear PDE with two underlying variables x and y :

$$A(x,y,u) \frac{\partial u}{\partial x} + B(x,y,u) \frac{\partial u}{\partial y} = C(x,y,u) \quad . \tag{5}$$

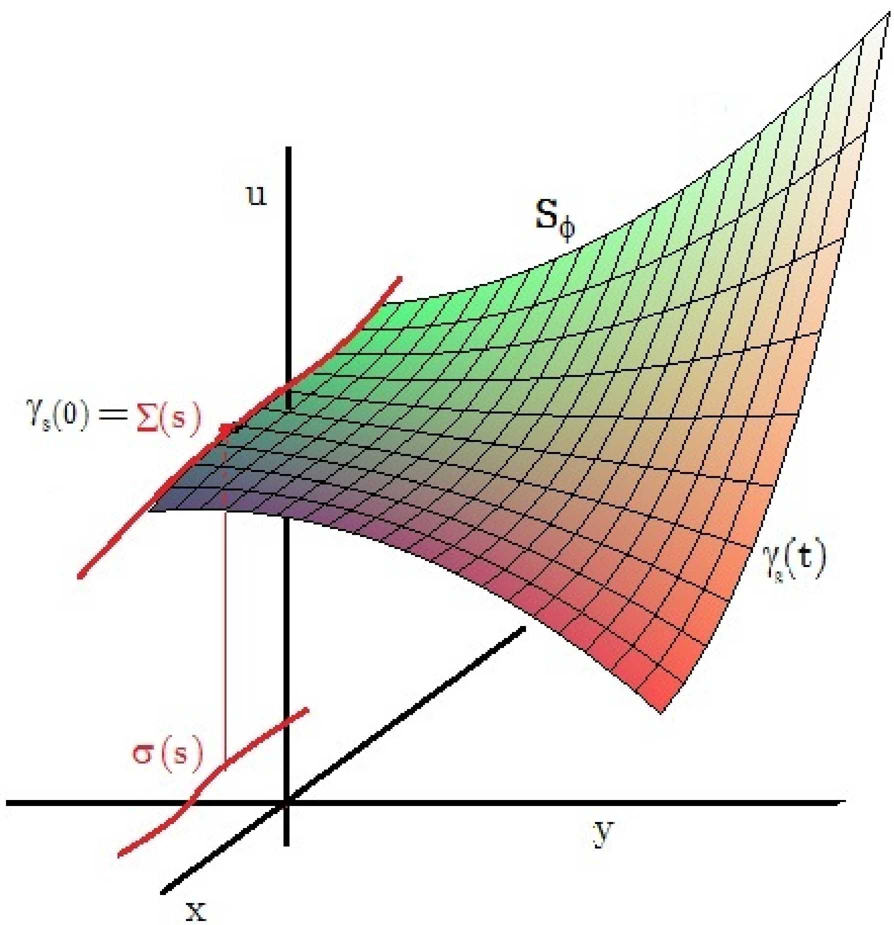
We shall suppose further that along with the PDE we are given a boundary condition along a certain curve $\boldsymbol{\sigma}(s)$ in the xy-plane; that is to say, a curve $\boldsymbol{\sigma}(s) = [x(s), y(s)] \in \mathbb{R}^2$ along which the values of our solution $\phi(x, y)$ fo the PDE (3) are specifed.

$$\phi(x(s), y(s)) = f(s) \tag{6}$$

Because we know the value of the solution ϕ at each point of $\boldsymbol{\sigma}(s)$, by setting $u(s) = \phi(x(s), y(s))$, we can lift $\boldsymbol{\sigma}(s)$ to a curve $\boldsymbol{\Sigma}(s)$ in the solution surface S_ϕ. Each value of the parameter s now corresponds to a point $\boldsymbol{\Sigma}(s)$ on the solution surface S_ϕ. On the other hand, once we specify a point in $\mathbb{R}^3$ we have a unique characteristic $\boldsymbol{\gamma}$ that passes through that point at $t = 0$. Let $\boldsymbol{\gamma}_s(t)$ be the characteristic that passes through $\boldsymbol{\Sigma}(s)$ at $t = 0$. Since this solution starts out on S_ϕ, the preceding theorem tells us it stays on the surface S_ϕ for all t. Thus, if we write

$$\boldsymbol{\gamma}_s(t) = [x_s(t), y_s(t), u_s(t)]$$

then the third component $u_s(t)$ is precisely $\phi(x_x(t), y_s(t))$, i.e. the value of the solution of the original PDE at the point $x = x_s(t)$, $y = y_s(t)$. Below is a picture of the situation we have just set up.



Unfortunately, as things stand right now, $u_s(t)$ is a function of s and t rather than our original coordinate variables x and y. However, we can utilize the conditions

$$\begin{aligned} x &= x_s(t) \\ y &= y_s(t) \end{aligned} \tag{7}$$

to solve for x and y in terms of s and t. So suppose we solve equation (7) for s and t, to get functions $s(x,y)$ and $t(x,y)$, expressing the parameters s and t as functions of x and y. We then have

$$\phi(x,y) = u_t(s) = u_{t(x,y)}(t(x,y))$$

and are thus able to express the value of the solution to the PDE as a function of x and y.

Example 3.5. Consider the following PDE/BVP:

$$\begin{aligned} x\phi_x + y\phi_y &= 1 + y^2 \\ \phi(x,1) &= x + 1 \end{aligned} \tag{8}$$

In this example we have

$$\begin{aligned} n &= 2 \\ F_1 &= x \\ F_2 &= y \\ G &= 1 + y^2 \\ \boldsymbol{\sigma}(s) &= \left\{[s,1] \in \mathbb{R}^2 \mid s \in \mathbb{R}\right\} \end{aligned} \tag{9}$$

Step 1. We'll first determine the characteristics for the PDE. These will be the solutions of

$$\begin{aligned} \tfrac{dx}{dt} &= x \\ \tfrac{dy}{dt} &= y \\ \tfrac{du}{dt} &= 1 + y^2 \end{aligned} \tag{10}$$

The first two equations are easily integrated to yield, respectively,

$$\begin{aligned} x(t) &= x_0 e^t \\ y(t) &= y_0 e^t \end{aligned}$$

Having determined the general form of $y(t)$ we can also integrate the third equation in (10) to yield

$$u(t) = \int \left(1 + y(t)^2\right) dt + C = \int \left(1 + y_0^2 e^{2t}\right) dt + C = t + \frac{1}{2} y_0^2 e^{2t} + C$$

Thus, the characteristics for our PDE are curves of the form

$$\boldsymbol{\gamma}_{x_0,y_0,C}(t) = \left[x_0 e^t, y_0 e^t, t + \frac{1}{2} y_0^2 e^{2t} + C\right] \tag{11}$$

Step 2. We now will try to determine which of the characteristics γ_{x_o} will pass over a given point (x,y). The first thing to do write down the lift $\boldsymbol{\Sigma}(s)$ of the curve $\boldsymbol{\sigma}(s)$ to the solution surface:

$$\Sigma(s) = [x(s), y(s), \phi(x(s), y(s))] = [s, 1, \phi(s,1)] = [s, 1, s+1] \tag{12}$$

We want a characteristic $\boldsymbol{\gamma}_{x_0,y_0,C}(t)$ to pass through $\boldsymbol{\Sigma}(s)$ at $t = 0$. Thus, we set

$$[s, 1, s+1] = \boldsymbol{\Sigma}(s) = \boldsymbol{\gamma}_s(0) = \left[x_0, y_0, 0 + \frac{1}{2} y_0^2 + C\right] \tag{13}$$

or, equating components on the extreme sides and solving for x_0, y_0 and C,

$$\begin{aligned} x_0 &= s \\ y_0 &= 1 \\ C &= s - \frac{1}{2} \end{aligned}$$

Thus, the particular characteristic that passes through $\boldsymbol{\Sigma}(s)$ at $t = 0$ is

$$\boldsymbol{\gamma}_s(t) = \boldsymbol{\gamma}_{s,1,s-\frac{1}{2}}(t) = \left[s e^t, e^t, t + \frac{1}{2} e^{2t} + s + \frac{1}{2}\right] \tag{14}$$

Step 3. We now choose $\boldsymbol{\gamma}_s(t)$ so that it passes over a point $[x, y]$ in the solution domain. Thus, setting

$$\begin{aligned} x &= se^t \\ y &= e^t \end{aligned}$$

and solving for x_o and t we get

$$\begin{aligned} t &= \ln|y| \\ s &= e^{-t}x = e^{-\ln|y|}x = \tfrac{x}{y} \end{aligned} \tag{15}$$

Step 4. We now set

$$\begin{aligned} \phi(x,y) &= (\boldsymbol{\gamma}_s)_u(t) \\ &= t + \frac{1}{2}e^{2t} + x_o + \frac{1}{2} \\ &= \ln|y| + \frac{1}{2}e^{2\ln|y|} + \frac{x}{y} + \frac{1}{2} \\ &= \ln|y| + \frac{1}{2}y^2 + \frac{x}{y} + \frac{1}{2} \end{aligned}$$

Thus, the solution to our PDE/BVP is

$$\boxed{\phi(x,y) = \ln|y| + \frac{1}{2}y^2 + \frac{x}{y} + \frac{1}{2}}$$

Example 3.6. Solve the following Cauchy problem.

$$\phi_x + \phi\phi_y = 0 \tag{16}$$

$$\phi(x,1) = x \tag{17}$$

in the region $x > 0$.

The differential equation for the characteristics is

$$\begin{aligned} \tfrac{dx}{dt} &= 1 \\ \tfrac{dy}{dt} &= u \\ \tfrac{du}{dt} &= 0 \quad . \end{aligned} \tag{18}$$

The first and last equations are the differential equations are easily solved:

$$x(t) = t + c_1 \quad , \tag{19}$$

$$u(t) = c_2 \tag{20}$$

Finally, using the $u(t)$ just found in the differential equation for $y(t)$, we get upon integration

$$y(t) = \int u(t)\,dt + c_3 = c_2t + c_3 \tag{21}$$

Thus, a characteristic for the PDE (16) is a curve of the form

$$\boldsymbol{\gamma}_{c_1,c_2,c_3}(t) = [t + c_1, c_2t + c_3, c_2]$$

The boundary values for this problem are specified along a line $\boldsymbol{\sigma}(s)$ in the xy-plane where $y = 1$:

$$\boldsymbol{\sigma}(s) = [s, 1]$$

and since $\phi(x, 0) = x$, we have

$$\boldsymbol{\Sigma}(s) = [s, 1, s]$$

So, if we demand that a particular characteristic $\boldsymbol{\gamma}_{c_1,c_2,c_3}(t)$ passes through the point $\Sigma(s)$ at $t = 0$, then we'll need

$$\begin{aligned} s &= 0 + c_1 \\ 1 &= c_2 \cdot 0 + c_3 = c_3 \\ s &= c_2 \end{aligned}$$

Solving these three equations for c_1, c_2 and c_3, we find

$$\begin{aligned} c_1 &= s \\ c_2 &= s \\ c_3 &= 1 \end{aligned}$$

Thus, the characteristic passing through $\boldsymbol{\Sigma}(s)$ at $t = 0$ is

$$\boldsymbol{\gamma}_s(t) = \boldsymbol{\gamma}_{s,1,0}(t) = [t + s, st + 1, s]$$

Finally, if we set

$$\begin{aligned} x &= t + s \\ y &= st + 1 \end{aligned}$$

If we use the first equation to substitute $x - s$ for t in the second equation we get

$$y = s(x - s) + 1$$

or

$$s^2 - sx + y - 1 = 0$$

Regarding this last equation as a quadratic equation for s and applying the Quadratic Formula we get

$$s = \frac{x \pm \sqrt{x^2 - 4(y-1)}}{2}$$

Now the condition

$$\boldsymbol{\gamma}_s(t) = [x, y, \phi(x, y)]$$

tells us that

$$\phi(x, y) = s = \frac{x \pm \sqrt{x^2 - 4(y-1)}}{2}$$