LECTURE 1

Introduction

1. Partial Differential Equations: Basics

A **partial differential equation** (or PDE, for short) is an equation relating a function ϕ of n variables $x_1, \ldots, x_n$, its partial derivatives with respect to the variables $x_1, \ldots, x_n$, and the variables themselves; that is to say, an equation of the form

$$F\left[\phi, x_1, \ldots, x_n, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \frac{\partial^2 \phi}{\partial x_1^2}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots\right](x) = 0 \tag{1}$$

The **order** of the partial differential equation is the order of the highest derivative appearing in the PDE.

If the dependence of the functional F on ϕ and its partial derivatives is linear, then the PDE (1) is said to be **linear**. In this introductory course we shall concentrate primarily on linear PDEs.

Remark 1.1. We might point out that there are several common conventions for expressing partial derivatives. First of all, it is especially common in physical applications to denote the underlying variables as x, y, z (or perhaps some other letters) rather than x_1, x_2, x_3. Secondly, it is commong to employ following shorthand expressions for partial derivatives

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \partial_x \phi = \phi_x \\ \frac{\partial^2 \phi}{\partial x \partial y} &= \partial_x \partial_y \phi = \phi_{xy} \\ &\quad etc. \end{aligned}$$

In these notes, we shall often pass back and forth between these various notations without comment.

2. Linear PDEs

A function f of one-variable x is **linear** if it can be expressed in the form

$$f(x) = ax + b$$

More generally, a function F of several variables $x_1, \ldots, x_n$ is (**simultaneously**) **linear with respect to** $x_1, \ldots, x_i$ if can be expressed as

$$F(x_1, \ldots, x_n) = A_1(x_{i+1}, \ldots, x_n)\, x_1 + A_2(x_{i+1} + \cdots + x_n)\, x_2 + \cdots + A_i(x_{i+1}, \ldots, x_n)\, x_i + B(x_{i+1}, \ldots, x_n)$$

or put another way

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} F = 0 \qquad \text{for all } k, \ell \in \{1, \ldots, i\}$$

Definition 1.2. *A partial differential equation*

$$F\left[\phi, x_1, \ldots, x_n, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \frac{\partial^2 \phi}{\partial x_1^2}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots\right] = 0$$

is said to be ***linear*** *if the function F that defines it is simultaneously linear with respect to* ϕ, $\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \frac{\partial^2 \phi}{\partial x_1^2}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots$.

N.B. A linear partial differential equation may be non-linear with respect to the underlying variables $x_1, \ldots, x_n$; however, it must be simultaneously linear with respect to the unknown function ϕ and all of its partial derivatives. In other words, a linear PDE will have the form

$$0 = A_0(x_1, \ldots, x_n)\,\phi + \sum_{i=1} A_i(x_1, \ldots x_n)\frac{\partial \phi}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(x_1, \ldots, x_n)\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \cdots$$

2.1. Differential operators.

2.1. Differential operators. Linear PDEs are often expressed in terms of **differential operators**. This means thinking of the function F that defines the PDE as a sum of terms of the form

$$A_{i_1 \cdots i_k}(x_1, \ldots, x_n)\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial \phi}{\partial x_{i_k}}$$

and then regarding such terms as arising via the application of the "monomial differential operator" of the form

$$A_{i_1 \cdots i_k}(x_1, \ldots, x_n)\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$$

to the function ϕ. The differential operator corresponding to F is then just the formal sum of the differential operators of the individual terms appearing in F.

Example 1.3.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + xy\phi = 0$$

is a linear PDE of degree 2. In differential operator notation, we might express this PDE as

$$\mathcal{L}[\phi] = 0$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}[\phi] := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + xy\right)\phi$$

Example 1.4.

$$x_1 x_2 \frac{\partial^2 \phi}{\partial x_1^2} + x_2 \frac{\partial \phi}{\partial x_2} + x_1^2 \phi - x_1^2 - x_2^2 = 0$$

is a linear PDE of degree 2. If we define the differential operator $\mathcal{L}$ to be the operation that sends a differentiable function ϕ to

$$\mathcal{L}[\phi] := x_1 x_2 \frac{\partial^2 \phi}{\partial x_1^2} + x_2 \frac{\partial \phi}{\partial x_2} + x_1^2 \phi$$

In this case, one would write

$$\mathcal{L} = x_1 x_2 \frac{\partial^2}{\partial x_1^2} + x_2 \frac{\partial}{\partial x_2} + x_1^2$$

then we can re-express the above differential equation as

$$\mathcal{L}[\phi] = x_1^2 + x_2^2 \quad .$$

2.2. More on linear PDEs.

Remark 1.5. The condition that a PDE

$$\mathcal{L}[\phi] = g(x_1, \ldots, x_n)$$

be linear amounts to the following condition on the differential operator $\mathcal{L}$

$$\mathcal{L}[c_1\phi + c_2\psi] = c_1\mathcal{L}[\phi] + c_2\mathcal{L}[\psi] \qquad \text{for all constants } c_1, c_2 \text{ and all differentiable functions } \phi \text{ and } \psi$$

Equivalently, a PDE $\mathcal{L}[\phi] = g(x_1, \ldots, x_n)$ is linear if the operator $\mathcal{L}$ acts a *linear transformation* on the vector space of differentiable functions on $\mathbb{R}^n$.

REMARK 1.6. Just as when linear differential equations expressed in terms of differential operators, it is common to distinguish PDEs of the form

$$\mathcal{L}[\phi] = 0$$

from those of the form

$$\mathcal{L}[\phi] = g(x) \neq 0$$

The former are called **homogeneous PDEs** and the latter **inhomogeneous PDEs.**

By a **solution** of the PDE (1) in a region $R \subset \mathbb{R}^n$, we mean an explicit function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$F[x, \Phi, \partial_i \Phi, \partial_i \partial_j \Phi, \ldots, \partial_i \partial_j \cdots \partial_k \Phi](x)$$

vanishes identically at each point $x \in R$. Note that if (1) has degree d then Φ must be of class C^d (i.e., Φ and each of partial derivatives up to order d must be continuous throughout R).

THEOREM 1.7 (Superposition Principle). *If ψ_1 and ψ_2 are two solutions of a linear homogeneous PDE $\mathcal{L}[\phi] = 0$, then any function of the form*

$$\psi = c_1 \psi_1 + c_2 \psi_2$$

is also a solution of $\mathcal{L}[\phi] = 0$.

THEOREM 1.8. *If Ψ is a solution of a inhomogeneous linear PDE*

$$\mathcal{L}[\phi] = g$$

and ψ is any solution of the corresponding homogeneous linear PDE

$$\mathcal{L}[\phi] = 0$$

Then $\Psi'(\mathbf{x}) = \Psi(\mathbf{x}) + \psi(\mathbf{x})$ is also a solution of the original inhomogeneous linear PDE.

3. Review of ODEs

Before undertaking our study of partial differential equations, let's take a minute to review some of the theory of ordinary differential equations.

Let's begin with the simplest possible ODE:

$$\frac{dx}{dt} = 0 \tag{2}$$

because this already tells us something important. This equation says that the rate of change of x with respect to t is 0. In other words,

$$\frac{dx}{dt} = 0 \quad \Rightarrow \quad x(t) = \text{ a constant.}$$

But the differential equation itself does not tell us what the constant is; indeed it can be any constant. One customarily writes the solution to

$$\frac{dx}{dt} = 0$$

as

$$x(t) = c \tag{3}$$

and thinks of the constant c as a parameter that once fixed specifies a particular solution of $\frac{dx}{dt} = 0$. The right hand side of (3) (with c interpreted as a variable parameter) is referred to as the *general solution* of (2).

As a second simple example of an ODE consider

$$\frac{dx}{dt} = f(t) \tag{4}$$

where $f(t)$ is some (continuous) function of t. The Fundamental Theorem of Calculus tells us that

$$x(t) - x(t_0) = \int_{t_0}^{t} \left(\frac{dx}{dt}\right) dt = \int_{t_0^t}^{t} f(t)\, dt = F(t) - F(t_0)$$

where $F(t)$ is any anti-derivative of $f(t)$ (i.e. $F(t)$ is any function whose derivative is $f(t)$; which is, of course is calculated by integrating $f(t)$ using various formulas for integration). If we set

$$F(t) = \int f(t)\, dt$$

and

(5) $$C = x(t_0) - F(t_0) \qquad (\textit{in toto} \text{ some constant})$$

then we have

(6) $$x(t) = \int f(t)\, dt + C$$

as the general solution of (4). As in the preceding example, our main point is that the general solution (from which all other solutions are a specialization) involves a single arbitrary constant.

More generally,

Theorem 1.9. *(Existence and Uniqueness Theorem for 1^{st} order ODEs). Suppose $F : \mathbb{R}^2 \to R$ is such that both $F(x,t)$ and $\frac{\partial F}{\partial x}(x,t)$ are differentiable functions of x and t around a point $(t_0, x_0) \in \mathbb{R}^2$. Then there exists one and only one solution to the ODE*

$$\frac{dx}{dt} = F(x,t)$$

such that

$$x(t_0) = x_0 \quad .$$

This theorem gives not only a simple check for the existence of solutions (we can expect solutions that are valid around all points (x,t) where F and $\frac{\partial F}{\partial x}$ exist and are differentiable) but it also tells us the (the graphs of) solutions can only cross each other at singular points (where the hypothesis of the theorem does not hold). Fixing an initial value t_0 of t, we can then parameterize the solutions valid around $t = t_0$ by the value of x at t_0.

$$x_0 \in \mathbb{R} \qquad \Longleftrightarrow \qquad \text{unique solution } \phi(t) \text{ of } \frac{dx}{dt} = F(x,t) \text{ with } \phi(t_0) = x_0$$

Another way of putting this is that the general solution of a first order ODE should involve a single parameter. However, we quickly add that specifying the value of a solution at a particular point, is just one way of selecting a particular solution. More often, in practice, the parameters which appear in the general solution of differential equation arise as constants of integration that appear in the course of undoing derivatives (as in the solution (6) of (4)). .

Even more generally, if

(7) $$\frac{d^n x}{dt^n} = F\left(t, x, \frac{dx}{dt}, \cdots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

is an n^{th} order differential equation, it's general solution (near generic points) will involve n arbitary constants, because, loosely speaking, there will be n derivatives to undo, and so n arbitary constants of integration in the solution. If the ODE (7) is in fact a linear ODE, then the solution space will, in fact, be an n-dimensional vector space.

4. Linear ODEs

Let me next recall some basic formulas for handling linear ordinary differential equations.

4.1. First order linear ODEs. The most general form of a first order linear ordinary differential equation is

$$a(x)\,y' + b(x)\,y + c(x) = 0$$

However, the formulas I give below for solutions of such an equation require one to first recast the such a first order linear ODE in *standard form*

$$y' + p(x)\,y = g(x) \tag{8}$$

where

$$p(x) \equiv \frac{b(x)}{a(x)} \quad , \quad g(x) \equiv -\frac{c(x)}{a(x)}$$

We say such a differential equation is *homogeneous* if $g(x) = 0$; i.e., the differential equation can be written

$$y' + p(x)\,y = 0 \tag{9}$$

The general solution to (9) is

$$y(x) = Cy_0(x)$$

where C is a constant and

$$y_0(x) \equiv \exp\left(-\int p(x)\,dx\right) \tag{10}$$

Note that (9) is a homogenous linear equation and that (10) says that the solution space is 1-dimensional (consisting of the scalar multiples of a single "vector").

Recall also the special solution (10) of (9) is also used to construct the the general solution (the more general situation) (8) via

$$y' + p(x)\,y = g(x) \quad \Rightarrow \quad y(x) = y_0(x)\int \frac{g(x)}{y_0(x)}dx + Cy_0(x)$$

where $y_0(x)$ is given by (10) and C is again a constant.

So in both case (equations (9) and (8)) we have infinitely many solutions (because the constant C can be any number). But because we have only one arbitrary number entering our formulas for the general solution, we can say, a little more precisely, that we have a 1-parameter family of solutions.

To get a unique solution (say one that corresponds to a particular experimental situation), one must place an additional condition on the solution; typically in the form of an *initial condition*

$$y(x_0) = y_0$$

Such a condition furnishes one with an additional equation that can be used to solve for C, thereby removing the arbitrariness in the general solution, and leaving one with a unique solution.

4.2. Second order linear ODEs. The standard forn of a second order linear ODE

$$a(x)\,y'' + b(x)\,y' + c(x)\,y + d(x) = 0$$

is

$$y'' + p(x)\,y' + q(x)\,y = g(x) \tag{11}$$

Like the case of first order linear ODEs the general solution of (11) is calculable from solutions of the corresponding homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{12}$$

Unfortunately, there is no closed formula analogous to (10) for solutions of (12). However, if one can find (or even guess) one solution of (12) the general solution to (11) can be calculated. This goes as follows.

Given one solution $y_1(x)$ of (12) a second, independent solution of (12) can be calculated via the Reduction of Order formula

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(-\int p(x)\right) dx$$

In terms of y_1 and y_2 the general solution to (11) is then

$$y(x) = -y_1 \int \frac{y_2 g}{y_1 y_2' - y_1' y_2} dx + y_2 \int \frac{y_1 g}{y_1 y_2' - y_1' y_2} dx + c_1 y_1 + c_2 y_2$$

where c_1, c_2 are arbitrary constants. Thus we have a two parameter family of solutions to (11). To get a unque solution, one needs two additional conditions to fix the values of c_1 and c_2. This is typically done by fixing the value of y and its first derivative at a particular point:

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}$$