LECTURE 0

Review of Basic ODE Theory

Before getting started in this course on Partial Differential Equations, we should perhaps review some of the highlights of the prerequiste course on elementary Ordinay Differential Equations.

1. Classification of ODEs

1.1. Underlying Variables and Unknown Functions. An ordinary differential equation is a differential equation that involves only one underlying variable. For example

$$\frac{dy}{dx} + 2xy = e^x$$

is an ordinary differential equation. The variable x is the variable used to differentiate - we call x the *underlying variable.* The variable y is what is being differentiated in the equation; we refer to y as the *unknown function* in the differential equation. It is important to realize that deciding on whether a symbol represents the unknown function or the underlying variable depends on whether the symbol is being differentiated or acting as the variable with respect to another symbol is being differentiated. Thus, in

$$\frac{dx}{dy} + 2yx = e^y$$

x is the unknown function and y is the underlying variable.

1.2. The Order of a Differential Equation. The *order* of a differential equation is the degree of the highest derivative that occur in the differential equation. Thus

$$\frac{d^2y}{dx^2} + 2x^2y = x + y$$

is a 2^{nd} order diffential equation, while

$$\left(\frac{df}{dx}\right)^4 + x\left(\frac{d^3f}{dx^3}\right)\left(\frac{df}{dx}\right) + x^2 = 0$$

is a 3^{rd} order differential equation (as the highest derivative occuring is the factor $\left(\frac{d^3f}{dx^3}\right)$ that occurs in the second term on the left).

1.3. Linear vs. Nonlinear Differential Equations. A function f of a single variable is linear if can be written

$$f(x) = mx + b$$

for a suitable choice of constants m and b. (It is called linear since the graph of such a function is always a straight line). Another, equivalent way of saying an function is linear with respect to x is as follows

$$f(x) \text{ is linear w.r.t. } x \quad \Longleftrightarrow \quad \frac{d^2f}{dx^2}(x) = 0$$

This latter definition provides a nice succinct extension to functions of more than one variable:

$$f(x_1, \ldots, x_k, y_{k+1}, \ldots, y_n) \text{ is linear w.r.t. } x_1, \ldots, x_k \text{ if } \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \text{ for all } i, j \in \{1, \ldots k\}$$

Alternatively, generalizing the first condition above, $f(x_1,\ldots,x_k,y_{k+1},\ldots,y_n)$ is linear w.r.t. $x_1,\ldots,x_k$ if it can be written in the form

$$f(x_1,\ldots,x_k,y_{k+1},\ldots,y_n) = m_1(y_{k+1},\ldots,y_n)\,x_1+m_2(y_{k+1},\ldots,y_n)\,x_2+\cdots+m_k(y_{k+1},\ldots,y_n)\,x_k+b(y_{k+1},\ldots,y_n)$$

So now we can speak of a function of several variables being linear with respect to a subset of its variables. We need this idea to define a linear differential equation.

Definition 0.1. *A differential equation is* ***linear*** *if it is linear with respect to the unknown function and its derivatives.*

Example 0.2.

$$\frac{dy}{dx}+xy^2=0$$

is a non-linear differential equation because the term xy^2 is nonlinear in y and y is the unknown function.

Example 0.3.

$$\frac{dx}{dy}+xy^2=0$$

is a linear differential equation because the right hand side is linear with respect to x and $\frac{d^2x}{dy^2}$ (the unknown function and its derivative). The circumstance that equation is nonlinear with respect to the underlying variable is irrelevant (as far as calling the differential equation linear or non-linear).

Example 0.4.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}+xy\phi=0$$

is a non-linear (partial) differential equation because if we write the PDE as

$$F\left(\frac{d^2\phi}{\partial x^2},\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\phi,y,x\right)=0 \qquad \text{with} \qquad F(a,b,c,d,e):=a+bc+def$$

then

$$\frac{\partial^2F}{\partial\left(\frac{\partial\phi}{\partial x}\right)\partial\left(\frac{\partial\phi}{\partial y}\right)}=\frac{\partial^2F}{\partial b\partial c}=1\neq0$$

and so it is not simultaneously linear with respect to $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$.

2. First Order ODEs

2.1. Existence and Uniqueness Theorem.

Let us write

$$\frac{dy}{dx}=F(x,y)$$

for a prototypical first order ordinary differential equation.

Theorem 0.5. *Suppose $F(x,y)$ is continuous with respect to x and y in a neighborhood of a point (x_0,y_0) in the xy-plane; and moreover so also is $\frac{\partial F}{\partial y}(x,y)$. Then there exists one and only one solution of*

$$\begin{aligned}\frac{dy}{dx}&=F(x,y)\\ y(x_0)&=y_0\end{aligned}$$

valid on a neighborhood of (x_0,y_0).

The additional condition

$$y(x_0)=y_0$$

is known as an *initial condition* (at x_0) or *boundary condition*. Put another way, the Existence and Uniqueness Theorem tells us that so long as $F(y,x)$ is not pathological at (x_0,y_0) we always have a solution to the differential equation, and moreover we can uniquely parameterize the set of solutions by asserting the value of the solution at the point x_0.

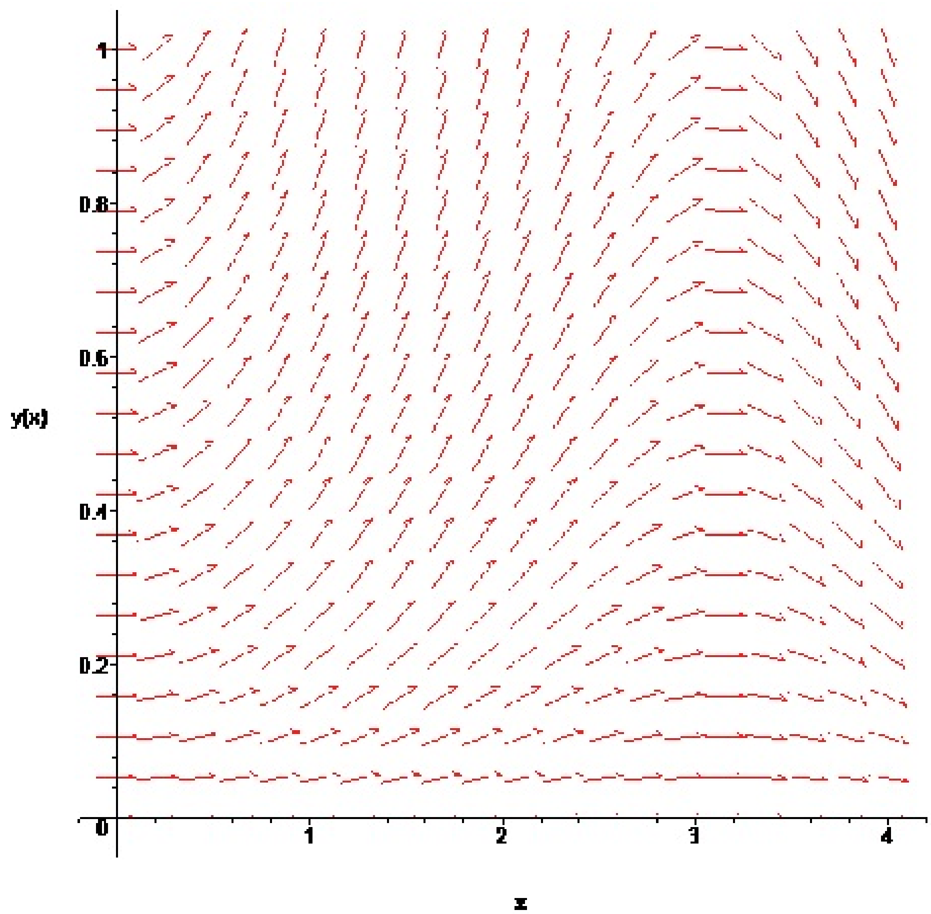
2.2. The Direction Field Picture. Recall that $\frac{dy}{dx}(x)$ is the slope to the tangent line to the graph of $y(x)$ at the point $(x, y(x))$. The differential equation

$$\frac{dy}{dx} = F(x, y)$$

thus has the effect of prescribing exactly how the graph of a solution passes through a point on the solution graph: if $\phi(x)$ is a solution and its graph passes through the point (x_0, y_0) then the tangent line to the graph of $\phi(x)$ at the point (x_0, y_0) must be $m = F(x_0, \phi(x_0)) = F(x_0, y_0)$.

A *direction field plot* of a differential equation $\frac{dy}{dx} = F(x, y)$ is formed by tabulating choosing a relatively fine grid of points (x_i, y_j), evaluating $F(x_i, y_j)$ at each of these points, and then drawing a short arrow with slope equal $F(x_i, y_i)$ at the point (x_i, y_i). What we're doing by this is figuring out how solutions of the differential equation must pass throught various points in the xy-plane. Here's an example of such a plot:

$$\frac{dy}{dx} = y \sin(x)$$



2.3. Solutions of First Order Linear ODEs. While there is no general formula for the solutions of

$$\frac{dy}{dx} = F(x, y)$$

there is a nice closed formula for the solutions of first order linear ODEs; i.e., differential equations of the form

$$y' + p(x)\,y = g(x)$$

The general solution of this equation is given by

$$y(x) = y_0(x) \int \frac{g(x)}{y_0(x)} dx + C y_0(x)$$

where

$$y_0(x) := \exp\left(-\int p(x)\, dx\right)$$

Here $y_0(x)$ is a general solution of

$$y' + p(x)\, y = 0.$$

2.4. Solutions of Nonlinear First Order ODEs. There are no closed formulas for solutions of this class of differential equations. There are, however, several techniques that can be applied if certain criteria are met. I won't review that material here; rather I simply recall the basic families of first order differential equations for which some specialized technique exists:

- Separable Equations: Differential equations of the form
$$M(x) + N(y) \frac{dy}{dx} = 0$$
- Exact Equations: Differential Equations of the form
$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \qquad \text{with } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
- Integrating Factors: Differential Equations of the form
$$M(x,y) + N(x,y) \frac{dy}{dx} = 0$$
where either
$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \quad \text{is independent of } y$$
or
$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \quad \text{is independent of } x$$
- Homogeneous Equations of Degree 0: Differential equations of the form
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

3. Second Order Linear ODEs

Definition 0.6. *A second order linear ordinary differential equation is a differential equation that can be cast in the form*

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x) \frac{dy}{dx} = g(x)$$

If $g(x) = 0$, the ODE is said to a ***homogeneous*** *linear ODE and if $g(x) \neq 0$ the differential equation is said to be* ***inhomogeneous****.*

Theorem 0.7. *If $p(x)$, $q(x)$ and $g(x)$ are all well-behaved near $x = x_0$, then there exists one and only one solution of*

$$\begin{aligned} y'' + p(x)\, y' + q(x)\, y &= g(x) \\ y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}$$

Note that a second order differential equation requires 2 initial conditions to fix a unique solution. In general, an n^{th} order ODE will require n additional conditions to fix a unique solution.

3.1. Homogeneous Case: $g(x) = 0$.

THEOREM 0.8 (Superposition Principle). *If y_1 and y_2 are solutions of*

$$y'' + p(x)\, y' + g(x)\, y = 0 \tag{1}$$

then so is any function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad , \qquad c_1, c_2 \text{ constants}$$

THEOREM 0.9. *If $y_1(x)$ and $y_2(x)$ are solutions of (1) such that there is no constant λ such that $y_2(x) = \lambda y_1(x)$, then every solution of (1) can be expressed as*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

DEFINITION 0.10. *If y_1, y_2 are two solutions of (1) and*

$$0 \neq W[y_1, y_2](x) \equiv y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x) \qquad \Longleftrightarrow \qquad y_2(x) \neq \lambda y_1(x)$$

then we say that y_1 and y_2 are two (linearly) ***independent*** *solutions of (1).*

THEOREM 0.11. *If y_1 is a solution of (1) then*

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(t))^2} \exp\left[\int^t -p(s)\, ds\right] dt$$

is a second linearly independent solution of (1).

Moral: In order to determine every solution of (1) it suffices to find one solution.

3.1.1. *Constant Coefficients Case.* Consider a second order linear ODE of the form

$$ay'' + by' + cy = 0 \tag{2}$$

with a, b, c constants. There always exists one solution of (2) of the form $y(x) = e^{\lambda x}$. Indeed, replacing y on the left hand side of (2) by $e^{\lambda x}$ we get

$$\alpha\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = \left(a\lambda^2 + b\lambda + c\right) e^{\lambda x}$$

which will vanish automatically if λ is a solution of

$$a\lambda^2 + b\lambda + c = 0 \tag{3}$$

Equation (3) is called the *characteristic equation* for (2). It is a quadratic equation whose solution can be determined via the Quadratic Formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three disctinct types of solution are possible

(i) $b^2 - 4ac > 0$. In this case we have two distinct real numbers λ, satisfying (3). Let's label them as λ_+ and λ_-. The general solution of (2) will be

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$$

(ii) $b^2 - 4ac = 0$. In this case, we have a unique real number $\lambda = \frac{-b}{2a}$ satisfying (3). It will furnish only one independent solution $y_1(x) = e^{-\frac{b}{2a}x}$ of (2). However, Theorem 0.11 can be applied to obtain a second independent solution and it will turn out to be $y_2(x) = xy_1(x) = xe^{-\frac{b}{2a}x}$. The general solution of (2) will be

$$y(x) = c_1 e^{-\frac{b}{2a}x} + c_2 x e^{-\frac{b}{2a}x}$$

(iii) $b^2 - 4ac < 0$. In this case, we will have two complex numbers as solutions of (3). Let us write these as $\lambda_\pm = \alpha \pm i\beta$. We have two linearly independent, complex-valued functions $\widetilde{y}_1(x) = e^{(\alpha+i\beta)x}$ and $\widetilde{y}_2(x) = e^{(\alpha-i\beta)x}$ as solutions to (2). These two solution are in fact complex-conjugates of one another and we can obtain two independent real-valued solutions by taking linear combinations of $\widetilde{y}_1(x)$ and $\widetilde{y}_2(x)$. Setting

$$\begin{aligned} y_1(x) &= \frac{1}{2}(\widetilde{y}_1(x) + \widetilde{y}_2(x)) = e^{\alpha x}\cos(\beta x) \\ y_2(x) &= \frac{1}{2i}(\widetilde{y}_1(x) = \widetilde{y}_2(x)) = e^{\alpha x}\sin(\beta x) \end{aligned}$$

we can write the general solution of (2) as

$$y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_1 e^{\alpha x}\sin(\beta x)$$

3.2. Inhomogeneous Case.

THEOREM 0.12. *The general solution of*

$$y'' + p(x)y' + q(x)y = g(x) \tag{4}$$

can be expressed as

$$y(x) = y_p(x) + c_1 y_1(x) + c_2(x)$$

where $y_1(x)$ and $y_2(x)$ are two independent solutions of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

and $y_p(x)$ is a particular solution of (4).

THEOREM 0.13. *Suppose $y_1(x)$ and $y_2(x)$ are two independent solutions of*

$$y'' + p(x)y' + q(x)y = 0$$

Then the functions $y_p(x)$ defined by the formula

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)g(x)}{W[y_1,y_2](x)}dx$$

will be a solution of

$$y'' + p(x)y' + q(x)y = g(x) \quad .$$

REMARK 0.14. In view of Theorems 0.11, 0.13 and 0.12, to find the complete solution of (4), it suffices to find one solution of

$$y'' + p(x)y' + q(x)y = 0 \quad .$$