Math 4263
Homework Set 6

1. Show that a function $f(z) = u(z) + iv(z)$ of a complex variable $z = x + iy$ that satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad , \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

also has the property that both its real part $u(z)$ and its imaginary part $v(z)$ satisfy Laplace's equation: i.e.,

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$$

2. Let $g(x)$ be any piecewise continuous function on $\mathbb{R}$. Show directly from the definition, that the mapping $\phi_g : C_c^\infty \to \mathbb{R}$ given by

$$\phi_g(f) := \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

defines a distribution. It will be easy to show that ϕ_g defines a linear functional. The hard part will be to demonstrate that ϕ_g is continuous. To this end, show that if $\{f_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ converges uniformly to a function $f(x) \in C_c^\infty(\mathbb{R})$, then

$$\lim_{n\longrightarrow\infty} \phi_g(f_n) = \phi_g(f)$$

By the way, *uniform convergence* means the following

- $\{f_n\}$ converges uniformly to f if for every $\varepsilon > 0$ there exists a natural number such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$ and all $n > N$.

3. Let ψ be any distribution. Show that the functional ψ' defined by

$$\psi'(f) := \psi\left(\frac{df}{dx}\right)$$

is a distribution.

4. Let $u(\mathbf{x}) = u(x,y)$ be a solution of Laplace's equation $\boldsymbol{\nabla}^2 u(\mathbf{x}) = 0$ on a planar domain D.

(a) Show that the function

$$f(\mathbf{x}) = \ln \|\mathbf{x}\|$$

is a solution of Laplace's equation on $\mathbb{R}^2$ except at $\mathbf{x} = \mathbf{0}$. (Hint: change to polar coordinates.)

(b) Let $f_\varepsilon(\mathbf{x}) = \ln(\|\mathbf{x}\| + \varepsilon)$. Show that

(i) $\lim_{\varepsilon\to 0} \boldsymbol{\nabla}^2 f_\varepsilon(\mathbf{x}) = 0$ whenever $\mathbf{x} \neq \mathbf{0}$.

(ii) $\int_{\mathbb{R}^2} \boldsymbol{\nabla}^2 f_\varepsilon(\mathbf{x})\, dA = 2\pi$ independent of ε

and conclude that

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi} \boldsymbol{\nabla}^2 \ln(\|\mathbf{x}\| + \varepsilon) = \delta^{(2)}(\mathbf{x})$$

(the 2-dimensional delta functional).

(c) Use Green's Identity

$$\int_D \phi \boldsymbol{\nabla}^2 \psi dA = \int_D \psi \boldsymbol{\nabla}^2 \phi dA + \int_{\partial D} (\phi \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\phi) \cdot \mathbf{n} ds$$

and the results of (b) to derive the *representation formula*

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \int_{\partial D} [u(\mathbf{x})(\boldsymbol{\nabla} \ln \|\mathbf{x} - \mathbf{x}_0\|) - (\boldsymbol{\nabla} u(\mathbf{x})) \ln \|\mathbf{x} - \mathbf{x}_0\|] \cdot \mathbf{n}\ ds$$

that expresses a solution u of Laplace's equation at an interior point $\mathbf{x}_0 \in D$ as a certain integral of u and its gradient over the boundary of D.