Math 4263
Homework Set 2

1. Use the Maximum Principle for the Heat Equation to demonstrate that there is a unique solution to

$$u_t - k^2 u_{xx} = f(x,t) \quad , \quad 0 \le x \le L \quad , \quad t > 0 \tag{1a}$$
$$u(0,t) = g(t) \quad , \quad t > 0 \tag{1b}$$
$$u(L,t) = h(t) \quad , \quad t > 0 \tag{1c}$$
$$u(x,0) = \phi(x) \quad , \quad 0 \le x \le L \tag{1d}$$

2. Prove the following identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2a}$$
$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \tag{2b}$$
$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2c}$$

3. Consider the following Heat Equation boundary value problem:

$$u_t - k^2 u_{xx} = 0 \quad , \quad 0 \le x \le L \quad , \quad t > 0 \tag{3a}$$
$$u(0,t) = 0 \quad , \quad t > 0 \tag{3b}$$
$$u(L,t) = 0 \quad , \quad t > 0 \tag{3c}$$
$$u(x,0) = \phi(x) \quad , \quad 0 \le x \le L \tag{3d}$$

(a) Apply the method of Separation of Variables to find a family of solutions of (3a) the form $u(x,t) = X(x)T(t)$.

(b) Impose the boundary conditions (3b) and (3c) to find a more specialized family of solutions $u_n(x,t) = X_n(x)T_n(t)$ satisfying (1a)–(1c).

(c) Set

$$u(x,t) = \sum_n a_n u_n(x,t)$$

where the $u_n(x,t)$ are the solutions found in (b), impose (3d), and then use properties of Fourier expansions to determine the coefficients a_n.

4. Find the solution of the following PDE/BVP:

$$u_t - u_{xx} = 0 \quad , \quad 0 \le x \le 1 \quad , \quad t > 0 \tag{4a}$$
$$u(0,t) = 0 \quad , \quad t > 0 \tag{4b}$$
$$u(1,t) = 0 \quad , \quad t > 0 \tag{4c}$$
$$u(x,0) = 1 - x^2 \quad , \quad 0 \le x \le 1 \tag{4d}$$