Math 4233
Solutions to Homework Set 4

1. For each of the following systems carry out the following steps.

(i) Identify the critical points.
(ii) For each critical point c, identify the corresponding linear system. Write down the general solution of these linear systems and discuss the stability of the solutions near the critical solution \( x(t) = c \).
(iii) Plot the direction field of the original system and discuss the evolution of the system for various initial conditions.

(a) \[
\begin{align*}
\frac{dx}{dt} & = x(1 - x - y) \\
\frac{dy}{dt} & = y(1.5 - y - x)
\end{align*}
\]

• Here we have

\[
F(x, y) = \begin{bmatrix} x(1 - x - y) \\ y(1.5 - y - x) \end{bmatrix} \Rightarrow
0 = F(x, y) \Rightarrow \begin{cases} x = 0, \ y = 0 \\ x = 0, \ y = 1.5 \\ x = 1, \ y = 0 \end{cases}
\]

Thus, there are only three critical points. We have

\[
\frac{dF}{dx} := \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 - 2x - y & -x \\ -y & 1.5 - x - 2y \end{bmatrix}
\]

– Linear system at \((0, 0)\):

\[
y' = Ay
\]

where

\[
A = \frac{dF}{dx} \bigg|_{(0, 0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}
\]

The matrix \(A\), being diagonal, obviously has eigenvalues 1 and 1.5 and has \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\) corresponding eigenvectors. Thus, the general solution of the linearization about \((0, 0)\) will look like

\[
y(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{1.5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Since both eigenvalues are positive, the origin will be an unstable critical point.

– Linear system at \((0, 1.5)\):

\[
A = \begin{bmatrix} -0.5 & 0 \\ -1.5 & -1.5 \end{bmatrix}
\]

eigenvalues/eigenvectors \(\rightarrow\) \(\begin{cases} r_1 = -0.5, \ \xi_1 = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \\ r_2 = -0.5, \ \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}\)

Since both eigenvalues are negative, the critical point at \((0, 1.5)\) is asymptotically stable.
Linear system at $(1, 0)$:

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 0.5 \end{bmatrix}$$

Eigenvalues/eigenvectors \( r_1 = -1, \quad \xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\( r_2 = -0.5, \quad \xi_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \)

Since the second eigenvalue is positive, the critical point at $(1, 0)$ is unstable.

Direction field plot

\[ \frac{dx}{dt} = x (1 - 0.5y) \]

\[ \frac{dy}{dt} = y (-0.25 + 0.5x) \]

- We have critical points at

\( (0, 0), \quad (0.5, 2) \)

and

\[ \frac{dF}{dx} = \begin{bmatrix} 1 - 0.5y & -0.5x \\ 0.5x & -0.25 + 0.5x \end{bmatrix} \]
- Linearized system at $(0, 0)$:

\[
A = \begin{bmatrix}
1 & 0 \\
0 & -0.25
\end{bmatrix}
\]

\[
y(t) = c_1 e^{t} \begin{bmatrix}
1 \\
0
\end{bmatrix} + c_2 e^{-0.25t} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

The critical point at $(0, 0)$ is thus unstable since the first eigenvalue $r_1 = 1$ is positive.

- Linearized system at $(0.5, 2)$:

\[
A = \begin{bmatrix}
0 & -0.25 \\
0.25 & 0
\end{bmatrix}
\]

eigenvalues/eigenvectors \rightarrow \begin{cases}
  r_1 = 0.25i, & \xi_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
  r_2 = -0.25i, & \xi_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}
\end{cases}
\]

\[
y(t) = c_1 e^{0.25it} \begin{bmatrix}
1 \\
-i
\end{bmatrix} + c_2 e^{-0.25it} \begin{bmatrix}
1 \\
-i
\end{bmatrix}
\]

\[
= c_1 \begin{bmatrix}
\cos(t/4) + i \sin(t/4) \\
-i \cos(t/4) + \sin(t/4)
\end{bmatrix} + c_2 \begin{bmatrix}
\cos(t/4) - i \sin(t/4) \\
\sin(t/4) - i \cos(t/4)
\end{bmatrix}
\]

Since the eigenvalues are both pure imaginary, analysis of this critical point by linearization is inconclusive.

- Direction field plot

2. For each of the following systems construct a suitable Liapunov function of the form $ax^2 + cy^2$ where $a$ and $c$ are to be determined. Then show that the critical point at the origin is of the indicated type.
(a) \[
\begin{align*}
\frac{dx}{dt} &= -x^3 + xy^2 \\
\frac{dy}{dt} &= -2x^2y - y^3
\end{align*}
\]
are asymptotically stable.

- We have
\[
\mathbf{F}(x, y) = \begin{bmatrix}
-x^3 + xy^2 \\
-2x^2y - y^3
\end{bmatrix}
\]

If
\[
\phi(x, y) = ax^2 + cy^2
\]
Then \(\phi(x, y)\) is positive-definite at the origin if
\[
0 < \left. \frac{\partial^2 \phi}{\partial x^2} \right|_{(0,0)} = 2a
\]
\[
0 < \left. \left( \frac{\partial^2 \phi}{\partial x^2} \right) \left( \frac{\partial^2 \phi}{\partial y^2} \right) - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right|_{(0,0)} = 4ac
\]
So we need both \(a, c > 0\). Now
\[
\dot{\phi} := \nabla \phi \cdot \mathbf{F} = (2ax) (-x^3 + xy^2) + (2cy) (-2x^2y - y^3)
\]
\[
= -2ax^4 + 2ax^2y^2 - 4cy^2x^2 - 2cy^4
\]
If we set \(a = 2c\), keeping \(c\) positive we have
\[
\dot{\phi}(x, y) = -4cx^4 - 2cy^4 = -2c(2x^4 + y^4)
\]
which is obviously negative definite since the factor \((2x^4 + y^4)\) is positive definite. Thus, we have identified a function \(\phi(x, y)\) which is positive definite at \((0, 0)\) and for which \(\nabla \phi \cdot \mathbf{F}\) is negative definite at \((0, 0)\). By the first Liapunov theorem, the autonomous system \(\dot{x} = \mathbf{F}(x)\) has a stable critical point at the origin.

(b) \[
\begin{align*}
\frac{dx}{dt} &= x^3 - y^3 \\
\frac{dy}{dt} &= 2xy^2 + 4x^2y + 2y^3
\end{align*}
\]
are unstable.

- Using the \(\phi(x, y) = ax^2 + cy^3\), we again have that \(\phi\) is positive definite about \((0, 0)\) so long as \(a, c > 0\). We also have
\[
\mathbf{F}(x, y) = \begin{bmatrix}
x^3 - y^3 \\
2xy^2 + 4x^2y + 2y^3
\end{bmatrix}
\]
\[
\Rightarrow \nabla \phi \cdot \mathbf{F} = (2ax) (x^3 - y^3) + (2cy) (2xy^2 + 4x^2y + 2y^3)
\]
\[
= 2ax^4 - 2ax^2y^3 + 4cy^3x + 8cy^2x^2 + 4cy^4
\]
\[
= 2ax^4 + (4c - 2a) xy^3 + 8cy^2x^2 + 4cy^4
\]
In order to demonstrate the instability of the critical point at \((0, 0)\) by applying the second Liapunov theorem, we need to identify a bounded domain in \(\mathbb{R}^2\) containing the origin where \(\phi(x, y)\) is positive definite and \(\nabla \phi \cdot \mathbf{F}\) is negative definite. Suppose we choose
\[
c = 1 \quad a = 2
\]
Then \(\phi(x, y)\) remains positive definite about \((0, 0)\), but now
\[
\nabla \phi \cdot \mathbf{F} = 4x^4 + 8x^2y^2 + 4y^4
\]
Note that the right hand side is a sum of non-negative terms, and so only vanishes if
\[
x^4 = 0 \text{ and } x^2y^2 = 0 \text{ and } y^4 = 0 \iff x = 0 \text{ and } y = 0
\]
Thus, $\nabla \phi \cdot \mathbf{F}$ is positive definite about $(0,0)$. By the second Liapunov theorem, the origin must be an unstable critical point (as we can choose the domain $D$ to any neighborhood of $(0,0)$).