Before applying various numerical methods, let’s write down the exact solution of
\[ x' = 2x - 3t \]
\[ x(0) = 1 \]
This is a first order, linear, non-homogeneous ODE with an initial condition. Such ODE/BVPs can be solved exactly as follows.
\[ x' + p(t)x = g(t) \quad , \quad x(0) = x_0 \]
\[ \implies x(t) = \frac{1}{\mu(t)} \int_0^t \mu(s) g(s) \, ds + \frac{x_0}{\mu(0)} \quad \text{where} \quad \mu(t) := \exp \left( \int_0^t p(s) \, ds \right) \]
In the case at hand,
\[ p(t) = -2 \quad , \quad g(t) = -3t \quad , \quad x_0 = 1 \]
and so we have
\[ \mu(t) = \exp \left( \int_0^t -2ds \right) = \exp \left( -2t \right) = e^{-2t} \]
\[ x(t) = \frac{1}{e^{-2t}} \int_0^t e^{-2s} (-3s) \, ds + \frac{1}{e^{-2t}} \]
\[ = e^{2t} \left( \frac{3}{2} e^{-2t} + \frac{3}{4} e^{-2t} - \frac{3}{4} \right) + e^{2t} \]
\[ = \frac{3}{2} t + \frac{3}{4} + \frac{1}{4} e^{2t} \]
The “exact” value of the solution at \( t = 0.4 \) is thus
\[ x(0.4) = 1.906385232 \]

1. Use the Euler method with a step size of 0.1 to determine an approximate value of the solution of
\[ x' = 2x - 3t \quad , \quad x(0) = 1 \]
at \( t = 0.4 \). Do the same using a step size of 0.01.

- The initial values \( t_0 = 0, \quad x_0 := x(t_0) = 1, \quad h = 0.1, \quad F(t, x) := 2x - 3t \), and the recursive algorithm
  \[ t_i = t_{i-1} + h \]
  \[ x_i = x_{i-1} + F(t_{i-1}, x_{i-1}) \, h \]
yield the following table

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 = 0 )</td>
<td>( x_0 = 1 )</td>
</tr>
<tr>
<td>( t_1 = 0.1 )</td>
<td>( x_1 = 1.20 )</td>
</tr>
<tr>
<td>( t_2 = 0.2 )</td>
<td>( x_2 = 1.410 )</td>
</tr>
<tr>
<td>( t_3 = 0.3 )</td>
<td>( x_3 = 1.6320 )</td>
</tr>
<tr>
<td>( t_4 = 0.4 )</td>
<td>( x_4 = 1.86840 )</td>
</tr>
</tbody>
</table>

So we conclude
\[ x(0.4) = x(t_4) \approx x_4 = 1.86840 \]

The percentage error in this case is
\[ \text{error} = \frac{x_{\text{exact}}(0.4) - x_4}{x_{\text{exact}}(0.4)} = 1.992527\% \]
• Repeating the computation using $h = 0.01$ (now requiring 40 steps), we obtain

$$x(0.4) = x(t_{40}) \approx x_{40} = 1.902009916$$

The percentage error in this case is

$$\text{error} = \frac{x_{\text{exact}}(0.4) - x_{40}}{x_{\text{exact}}(0.4)} = -0.29508\%$$

2. Use the Huen method (also known as the second order Runge-Kutta method) with a step-size of 0.01 to compute an approximate value for the solution of (1) at $t = 0.4$.

• The Huen method corresponds to the following algorithm:

$$t_i = t_{i-1} + h$$

$$x_i = x_{i-1} + \frac{1}{2} (F(t_{i-1}, x_{i-1}) + F(t_{i-1} + h, x_{i-1} + F(t_{i-1}, x_{i-1}) h)) h$$

or, inserting a couple intermediary calculations as a substeps,

$$t_i = t_{i-1} + h$$

$$f_1 = F(t_{i-1}, x_{i-1})$$

$$f_2 = F(t_{i-1} + h, x_{i-1} + hf_1)$$

$$x_i = x_{i-1} + \frac{h}{2} (f_1 + f_2)$$

This algorithm yields a table like

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.01</td>
<td>1.020050000</td>
</tr>
<tr>
<td>0.02</td>
<td>1.040202010</td>
</tr>
<tr>
<td>0.03</td>
<td>1.060458091</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.39</td>
<td>1.880340130</td>
</tr>
<tr>
<td>0.40</td>
<td>1.906356001</td>
</tr>
</tbody>
</table>

We conclude

$$x(0.4) = x(t_{40}) \approx x_{40} = 1.906356001$$

The percentage error in this case is

$$\% \text{ error} = \frac{x_{\text{exact}}(0.4) - x_{40}}{x_{\text{exact}}(0.4)} = -0.001533\%$$

3. Use the fourth order Runge-Kutta method with a step-size of 0.01 to compute an approximate value for the solution of (1) at $t = 0.4$.

• The iterative computations for the fourth order Runge-Kutta method are given by

$$t_i = t_{i-1} + h$$

$$f_{1,i-1} = F(t_{i-1}, x_{i-1})$$

$$f_{2,i-1} = F(t_{i-1} + h/2, x_{i-1} + hF_{1,i-1}/2)$$

$$f_{3,i-1} = F(t_{i-1} + h/2, x_{i-1} + hF_{2,i-1}/2)$$

$$f_{4,i-1} = F(t_{i-1} + h, x_{i-1} + hF_{3,i-1})$$

$$x_i = x_{i-1} + \frac{h}{6} (f_{1,i-1} + 2f_{2,i-1} + 2f_{3,i-1} + f_{4,i-1})$$
This leads to a table of the form

\[
\begin{align*}
t_0 &= 0.0 \quad x_0 = 1 \\
t_1 &= 0.01 \quad x_1 = 1.020050335 \\
t_2 &= 0.02 \quad x_2 = 1.040202694 \\
t_3 &= 0.03 \quad x_3 = 1.060459137 \\
t_4 &= 0.04 \quad x_4 = 1.080821767 \\
t_5 &= 0.05 \quad x_5 = 1.101292730 \\
\vdots & \quad \vdots \\
t_{38} &= 0.38 \quad x_{38} = 1.854569056 \\
t_{39} &= 0.39 \quad x_{39} = 1.880368067 \\
t_{40} &= 0.40 \quad x_{40} = 1.906385233 \\
\end{align*}
\]

The percentage error in this case is

\[ \% \text{ error} = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.4)} = 0.00000\% \]

4. Use the fourth order Adams-Bashforth multi-step method with a step-size of 0.01 to compute an approximate value for the solution of (1) at \( t = 0.4 \).

- The fourth order Adams-Bashforth uses the following recursive formula.

\[
t_i = t_{i-1} + h \\
x_i = x_{i-1} + \frac{h}{24} (55F(t_{i-1}, x_{i-1}) - 59F(t_{i-2}, x_{i-2}) + 37F(t_{i-3}, x_{i-3}) - 9F(t_{i-4}, x_{i-4}))
\]

To implement this we need four initial points. These we’ll take from the fourth order Runge-Kutta computation performed above. One obtains in this way a table of the form

\[
\begin{align*}
t_0 &= 0.0 \quad x_0 = 1 \\
t_1 &= 0.01 \quad x_1 = 1.020050335 \\
t_2 &= 0.02 \quad x_2 = 1.040202694 \\
t_3 &= 0.03 \quad x_3 = 1.060459137 \\
t_4 &= 0.04 \quad x_4 = 1.080821767 \\
t_5 &= 0.05 \quad x_5 = 1.101292730 \\
\vdots & \quad \vdots \\
t_{38} &= 0.38 \quad x_{38} = 1.854569037 \\
t_{39} &= 0.39 \quad x_{39} = 1.880368047 \\
t_{40} &= 0.40 \quad x_{40} = 1.906385212 \\
\end{align*}
\]

The percentage error in this case is

\[ \% \text{ error} = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.4)} = -.000001\% \]