Math 4233

SOLUTIONS TO FIRST EXAM

October 28, 2016

1. Find the general solution of the following homogeneous linear systems

(a) $\mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} \mathbf{x}$

- The eigenvalues of the coefficient matrix are

$$0 = \det \begin{bmatrix} -1-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} = \lambda^2 + 3\lambda - 4 = (\lambda+4)(\lambda-1) \quad \Longrightarrow \quad \lambda = 1, -4$$

eigenvector for $\lambda = 1$:

$$NullSp\left(\begin{bmatrix} -1-1 & 2 \\ 3 & -2-1 \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right) = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenvector for $\lambda = -4$:

$$NullSp\left(\begin{bmatrix} -1-(-4) & 2 \\ 3 & -2-(-4) \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} \right) \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We thus two linearly independent eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with eigenvalues, respectively $\lambda = 1$ and $\lambda = -4$. Each eigenvalue/eigenvector $(\lambda, \boldsymbol{\xi})$ pair corresponds to an independent solution of the form $e^{\lambda t}\boldsymbol{\xi}$. The general solution to the problem is thus

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

(b) $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

- The eigenvalues of the coefficient matrix are

$$0 = \det\left(\begin{bmatrix} -2-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \right) = (\lambda+2)^2 \quad \Longrightarrow \quad \lambda = -2$$

eigenvector for $\lambda = -2$

$$NullSp\left(\begin{bmatrix} -2-(-2) & 0 \\ 1 & (-2)-(-2) \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \Longrightarrow \quad \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since we have only one eigenvector we'll need to find a second *generalized eigenvector* $\boldsymbol{\eta}$ satisfying

$$(\mathbf{A} - \lambda\mathbf{I})\,\boldsymbol{\eta} = \boldsymbol{\xi} \quad \Longrightarrow \quad \begin{bmatrix} -2-(-2) & 0 \\ 1 & (-2)-(-2) \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The augmented matrix for this linear system is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \eta_1 = 1$, η_2 is free parameter. We can freely set the free parameter corresponding to η_2 equal to zero and take $\boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The general solution is of the system is then

$$\mathbf{x}(t) = c_1 e^{\lambda t}\boldsymbol{\xi} + c_2\left(te^{\lambda t}\boldsymbol{\xi} + \boldsymbol{\eta}\right) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\left(te^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

2. Consider the autonomous nonlinear system : $\dfrac{dx}{dt} = 1 - y \quad , \quad \dfrac{dy}{dt} = x^2 - y^2$

(a) Determine the critical points of this system,

The critical points for the system are the points $[x_0, y_0]$ at which the functions on the right hand side of the system equations vanish.

$$\left.\begin{array}{c} 0 = 1 - y \\ 0 = x^2 - y^2 \end{array}\right\} \quad \Longrightarrow \quad y = 1, x = \pm 1 \quad \Longrightarrow \quad \text{critical points at } (1,1), (-1,1)$$

(b) Determine the corresponding linear systems near these critical points and discuss the stability of solutions near these critical points.

- $$\frac{d\mathbf{F}}{d\mathbf{x}} = \left[\begin{array}{cc} 0 & -1 \\ 2x & -2y \end{array}\right]$$
  - At $(1,1)$, $\frac{d\mathbf{F}}{d\mathbf{x}} = \left[\begin{array}{cc} 0 & -1 \\ 2 & -2 \end{array}\right]$. This matrix has eigenvalues
    $$0 = \det\left(\left[\begin{array}{cc} 0-\lambda & -1 \\ 2 & -2-\lambda \end{array}\right]\right) = \lambda^2 + 2\lambda + 2 \quad \Longrightarrow \quad \lambda = \frac{-2 + \sqrt{4-8}}{2} = -1 \pm i$$
    Since both eigenvalues have negative real part, this critical point is asymptotically stable.
  - At $(-1,1)$, the linearization matrix is
    $$\left[\begin{array}{cc} 0 & -1 \\ -2 & -2 \end{array}\right]$$
    The characteristic equation for this matrix is
    $$0 = \det\left(\left[\begin{array}{cc} 0-\lambda & -1 \\ -2 & -2-\lambda \end{array}\right]\right) = -\lambda(-2-\lambda) - 2 = \lambda^2 + 2\lambda - 2 \quad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$
    Since $-1 + \sqrt{3}$ is a positive eigenvalue, the critical point at $(-1,1)$ is unstable.

3. Find the solution of the following heat conduction problem.

$$\begin{aligned} 4u_t - u_{xx} &= 0 \quad , \quad 0 < x < 2 \quad , \quad t > 0 \\ u(0,t) &= 0 \\ u(2,t) &= 0 \\ u(x,0) &= 4\sin(2\pi x) \end{aligned}$$

As you solve it, please try to explain each (conceptual) step that you take in solving the problem.

- We first use separation of variables to find a suitable family of solutions of the heat equation satisfying the first two boundary conditions. Thus, we look for functions of the form

(1e) $$u(x,t) = X(x)\,Y(t)$$

that will satisfy (1a), (1b) and (1c). Substituting (1e) into (1a) and dividing the result by $X(x)\,Y(t)$ we obtain

$$4\frac{\dot{Y}}{Y} = \frac{X''}{X}$$

Since the left hand side does not depend on x and the right hand side does not depend on t, we conclude that both sides must be equal to a constant, which we shall denote by $-\lambda^2$. We are thus led to

$$\begin{aligned} X'' = -\lambda^2 X \quad &\Longrightarrow \quad X(x) = A\sin(\lambda x + \delta) \\ \dot{Y} = -\frac{\lambda^2}{4}Y \quad &\Longrightarrow \quad Y(t) = Ce^{-\frac{\lambda^2}{4}t} \end{aligned}$$

Imposing the boundary conditions at $x = 0$ on the expression (1e) we find

$$0 = u(0,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(\delta) \quad \Longrightarrow \quad \delta = 0 \qquad \text{(for non-trivial solutions)}$$

Taking then $\delta = 0$ and imposing the boundary condition at $x = 2$ we find

$$0 = u(2,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(2\lambda) \quad \Longrightarrow \quad \lambda = \frac{n\pi}{2} \quad , \quad n = 1, 2, \ldots \qquad \text{(for non-trivial solutions)}$$

Thus, any function of the form

$$\phi_n(x,t) = e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) = e^{-\left(\frac{n\pi}{4}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) \quad , \quad n = 1, 2, \ldots$$

will satisfy equations (1a) - (1c). Moreover, any linear combination of the functions ϕ_n will continue to satisfy equations (1a) - (1c). We thus set

(1f) $$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{4}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) =$$

and try to choose the coefficients a_n so that the final boundary condition (1d) is satisfied. Plugging (1f) into (1d) we obtain

$$4\sin(2\pi x) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{4}\right)^2 0}\sin\left(\frac{n\pi}{2}x\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}x\right)$$

From this we conclude that the coefficients a_n should coincide with coefficients of $\sin\left(\frac{n\pi}{2}x\right)$ in the Fourier-sine expansion of the function on the right hand side on the interval $[0,2]$. Thus, we set

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L 4\sin(2\pi x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= 4\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi}{2}x\right)dx \end{aligned}$$

By the orthogonality properties of the Fourier-sine functions

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

we find

$$a_n = \begin{cases} 4 & \text{if } n = 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$u(x,t) = 4e^{-\left(\frac{4\pi}{4}\right)^2 t} \sin(2\pi x) = 4e^{-\pi^2 t} \sin(2\pi x)$$