LECTURE 20

A Vibrating Membrane Problem

In this lecture we work out in detail the solution to a particular PDE/BVP so that you see how all the mathematical apparatus developed over the last series of lectures is applied in practice.

PROBLEM 20.1. *Consider a circular drum head of radius b. The vibrations of this drum head are governed by the wave equation*

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1}$$

Suppose we have, in addition, a boundary condition and initial conditions of the form

$$\begin{aligned} \phi(b,\theta,t) &= 0 \\ \phi(r,\theta,0) &= f(r) \\ \frac{\partial \phi}{\partial t}(r,\theta,0) &= 0 \end{aligned} \tag{2}$$

(20.1)

What is the solution to this PDE/BVP?

In the course of solving this problem (which is also going to appear as a homework problem). We are going to run into a number of little obstacles that actually I'm happy to deal with. This because out in the wild, real world problems do not necessarily conform to nice succinct theorem statements. What I hope you get out of this exercise is the fortitude to try and modify general methods in order to get enough out of them to solve a particular problem.

1. Separation of Variables

We'll begin by constructing a large family of *relatively simple* solutions to the PDE. Since this problem has a natural circular symmetry, we'll write down the PDE in polar coordinates and then apply the Separation of Variables techique to get a nice family of solutions.

Thus, in polar coordinates the wave equation is

$$\frac{\partial^2 \phi}{\partial t^2} - c^2\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \tag{3}$$

We then pose the following ansatz for a solution of (3):

$$\phi(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t) \tag{4}$$

Plugging this ϕ into (3) and then dividing the resulting equation by $c^2 R(r)\,\Theta(\theta)\,T(t)$ we get

$$\frac{1}{c^2}\frac{T''}{T} - \frac{R''}{R} - \frac{1}{r}\frac{R}{R} - \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

or

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

By the usual Separation of Variables argument, both sides must equal a constant, which we will will denote by $-\lambda^2$. This allows us to morph the original PDE to a ODE plus a simpler PDE

$$T'' + c^2\lambda^2 T = 0 \tag{5}$$

and

$$\frac{R''}{R} + \frac{1}{r}\frac{R}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

If we multiply the second equation by r^2 we can get a further decoupling

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 = -\frac{\Theta''}{\Theta} \tag{6}$$

Now the ustual Separation of Varibles argument tells us that both sides must equal a constant, say α^2. We now can split (6) into a pair of weakly coupled ODEs:

$$\Theta'' + \alpha^2\Theta = 0 \tag{7}$$

and

$$r^2R'' + rR' + \lambda^2r^2R - \alpha^2R = 0 \tag{8}$$

In summary, anytime $T_\lambda(t)$ is a solution of (5), $\Theta_\alpha(\theta)$ is a solution of (7) and $R_{\alpha,\lambda}(r)$ is a solution of (8) ,

$$\phi_{\lambda,\alpha}(r,\theta,t) = R_{\alpha,\lambda}(r)\,\Theta_\alpha(\theta)\,T_\lambda(t) \tag{9}$$

will be a solution of (3).

2. Solutions of the ODEs

Solving (5) and (7) is pretty easy, since they are recognizable as the differential equation for Fourier sine and Fourier cosine functions, we have

$$T_\lambda(t) = a_\lambda\cos(c\lambda t) + b_\lambda\sin(c\lambda t)$$

and

$$\Theta_\alpha(\theta) = c_\alpha\cos(\alpha\theta) + d_\alpha\sin(\alpha\theta)$$

In view of the boundary conditions (2) imposed at $t = 0$, it makes sense to simply discard the solutions $\cos(c\lambda t)$ from the general solution $T_\lambda(t)$. On the other hand, because we expect our solutions to be periodic with respect to the angle parameter θ, we'll need

$$\Theta_a(\theta + 2\pi) = \Theta_\alpha(\theta) \quad \Rightarrow \quad \alpha = n \in \mathbb{N}$$

The requirement that the parameter α must an integer n is, of course, also going to change the differential equation for the radial function: equation (8) now becomes

$$r^2R'' + rR' + \lambda^2r^2R - n^2R = 0 \tag{10}$$

and alas, this is a more difficult differential equation. We shall tackle it none-the-less. But before doing so, let's look ahead a bit.

Once we find the solutions $R_{n,\lambda}$ to (10), the plan would be to use the following functional form

$$\phi(r,\theta,t) = \sum_{\lambda,n} c_{\lambda,n}R_{\lambda,n}(r)\,\Theta_n(\theta)\,T_\lambda(t)$$

as ansatz for a solution of (3) that satisfies the boundary conditions (2). To satisfy the second boundary condition we'll need

$$f(r) = \phi(r,\theta,0) = \sum_{\lambda,n} c_{\lambda,n}R_{\lambda,n}(r)\,\Theta_n(\theta)\,T_\lambda(0)$$

To satisfy this for all θ, we ill have to restrict $n = 0$ and takeing only the sine-type part of $T_\lambda(t)$. Then we will be left with choosing the coefficients $c_{\lambda,0}$ so that

$$f(r) = \sum_\lambda c_{\lambda,0}R_{\lambda,0}(r)$$

and then to determine the coefficients $c_{\lambda,0}$, we'll need some semblance of Sturm-Liouville theory.

However, the differential equation (10) is not quite of Sturm-Liouville form. Moreover, we can't expect a Sturm-Liouville type boundary condition to be valid at $r = 0$.

The first difficulty is easy to get around, if we divide equation (10) by r we get

$$rR'' + R' - \frac{n^2}{r}R + \lambda^2 rR = 0$$

or

$$\frac{d}{dr}\left(rR'\right) - \frac{n^2}{r}R + \lambda^2 rR = 0$$

which is of Sturm-Liouville form with $\widetilde{p}(r) = r$, $\widetilde{q}(r) = -n^2/r$ and $\widetilde{r}(r) = r$. However, to arrive at the orthogonality property of S-L eigenfunctions we needed besides an S-L differential equation, S-L boundary conditions of the form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned}$$

while in the current problem, we only have a boundary condtion of the form

$$R_{\lambda,0}(b) = 0.$$

3. An Orthogonality Property for Solutions of the Radial Equation

Let me quickly rehash how the orthogonality property of Sturm-Liouville eigenfunctions comes about. Let

$$L[y] = \frac{d}{dx}\left(\widetilde{p}(x)\frac{dy}{dx}\right) + q(x)y(x) = \lambda r(x)y$$

be a Sturm-Liouville differential equation.

THEOREM 20.1 (Lagrange's Identity). *If $\phi(x)$ and $\psi(x)$ satisfy boundary conditions of the form*

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(b) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned} \qquad (11)$$

then

$$\int_a^b \phi L[\psi]\,dx = \int_a^b \psi L[\phi]\,dx$$

Now suppose $\phi(x)$ and $\psi(x)$ are two Sturm-Liouville eigenfuctions with different eigenvalues, say

$$L[\phi] = \lambda_1\phi \quad m \quad L[\psi] = \lambda_2\psi \qquad \text{with } \lambda_1 \neq \lambda_2$$

If $\phi(x)$ and $\psi(x)$ also satisfy Sturm-Liouville type boundar conditions then Lagranges identity says

$$\int_a^b \phi(x)\left(\lambda_2 r(x)\psi(x)\right) = \int_a^b \psi(x)\left(\lambda_1 r(x)\phi(x)\right)dx$$

or

$$(\lambda_1 - \lambda_2)\int_a^b \phi(x)\psi(x)r(x)\,dx = 0$$

which implies

$$\int_a^b \phi(x)\psi(x)r(x)\,dx = 0$$

if $\lambda_1 \neq \lambda_2$.

In the case at hand, our differential

$$r^2R'' + rR' + \lambda^2 r^2 R - n^2 R = 0$$

is not quite of Sturm-Liouville type; and more problematically, we only have a boundary condition at one endpoint of the interval of interest $[0, b]$:

$$R(b) = 0$$

(which corresponds to keeping the drum head stationary at its perimeter).

Now the first discrepancy with the standard Sturm-Liouville theory is easy to resolve. Dividing the radial equation (10) by r yields

$$rR'' + R' - \frac{n^2}{r}R = -\lambda^2 rR$$

or

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = \lambda^2 rR$$

which is of Sturm-Liouville form.

As for the fact that we only have one boundary condition, we shall be saved by the fact that in our case the factor of r that occurs inside the derivative term of the S-L differential equation vanishes at $r = 0$. To see this, let me recall briefly the proof of the Lagrange Identity. One starts with

$$\int_a^b \phi(x)\left[\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + q(x)\psi(x)\right]dx$$

and then integrates twice by parts to get

$$\int_a^b \phi(x)\left[\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + q(x)\psi(x)\right]dx = \int_a^b \psi(x)\left[\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + q(x)\psi(x)\right]dx + \left(p(x)\phi(x)\psi'(x) - p(x)\phi'(x)\psi(x\right.$$

or

$$\int_a^b \phi L[\psi]\,dx - \int_a^b \psi L[\phi]\,dx = \left.\left(p(x)\phi(x)\psi'(x) - p(x)\phi'(x)\psi(x)\right)\right|_a^b \tag{12}$$

and **then** one applies the boundary conditions (11) to show that the right hand side of (12) vanishes, independently at both endpoints.

However, there is another way to get the right hand side of (12) to vanish at say $r = a$. For example if $p(a) = 0$, then $p(x)\phi(x)\psi'(x) - p(x)\phi'(x)\psi(x)$ automatically vanishes at $x = a$. We thus have the following modification of Lagrange's Identity:

Theorem 20.2 (Modified Lagrange Identity). *Suppose*

$$L[y] = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$$

with

$$p(0) = 0 \quad .$$

Then whenever ϕ, ψ are differentiable functions satisfying that vanish at $x = b$, we have

$$\int_0^b \phi L[\psi]\,dx = \int_0^b \psi L[\phi]\,dx$$

Noting that our radial solutions satisfy the hypotheses of the modified Lagrange identity; the relevant differential operator in this case is

$$\frac{d}{dr}\left[r\frac{dR}{dr}\right] - \frac{n^2}{r}R$$

and $R(r)$ is required to vanish at $r = b$.

Indeed, if we have two radial solutions R_{n,λ_1} and R_{n,λ_2} corresponding to different values of λ

$$\begin{aligned} L[R_{n,\lambda_1}] &= -\lambda_1^2 rR_{n,\lambda_1} \\ L[R_{n,\lambda_2}] &= -\lambda_2^2 rR_{n,\lambda_2} \end{aligned}$$

Then

$$\begin{aligned} 0 &= \int_0^b R_{n,\lambda_1} L\left[R_{n,\lambda_2}\right] dx - \int_0^b R_{n,\lambda_2} L\left[R_{n,\lambda_1}\right] dx \\ &= \int_0^b R_{n,\lambda_1}\left(-\lambda_2^2 r R_{n,\lambda_2}\right) dx - \int_0^b R_{n,\lambda_2}\left(-\lambda_1^2 r R_{n,\lambda_1}\right) dx \\ &= \left(\lambda_1^2 - \lambda_2^2\right)\int_0^b R_{n,\lambda_1} R_{n,\lambda_2} r dr \end{aligned}$$

which then requires

$$\int_0^b R_{n,\lambda_1} R_{n,\lambda_2} r dr = 0 \quad \Longleftrightarrow \quad \lambda_1^2 \neq \lambda_2^2$$

Conclusion 20.3. *Even though the solutions to the radial problem are not quite Sturm-Liouville eigenfunctions, they will nevertheless provide us with a nice family of orthogonal functions.*

4. Construction of the formal solution to the PDE/BVP

Let me review what we have gotten so far. We have a large family $\left\{\phi_{\lambda,\alpha}(r,\theta,t) = R_{\alpha,\lambda}(r)\,\Theta_\alpha(\theta)\,T_\lambda(t)\right\}$ of Separation of Variables type solutions to the PDE. A general solution would thus be of the form

$$\phi(r,\theta,t) = \sum_{\lambda,\alpha} c_{\alpha,\lambda} R_{\alpha,\lambda}(r)\,\Theta_\alpha(\theta)\,T_\lambda(t)$$

Here the functions $R_{\alpha,\lambda}(r)$, $\Theta(\theta)$, and $T_\lambda(t)$ are solutions to, respectively, the ODEs (8), (7) and (5). We don't need all these solutions, and we will immediately toss out the solutions that we don't need (or otherwise hinder us). Because there is no explicit θ-dependence in the problem statement, we should not expect at θ-dependent solution. Therefore, we should set the separation constant $\alpha = 0$ and keep only the constant solution

$$\Theta_0(\theta) = 1$$

This reduces our general ansatz to

$$\phi(r,\theta,t) = \sum_{\lambda} c_\lambda R_{0,\lambda}(r)\,T_\lambda(t)$$

Now consider the second boundary conditions at $t = 0$.

$$0 = \frac{\partial\phi}{\partial t}(r,\theta,0) = \sum_\lambda c_\lambda R_{0,\lambda}(r)\,T_\lambda'(0)$$

The simplest way to ensure this condition is to make sure $T_\lambda'(0) = 0$. But the T_λ are solutions of

$$T_\lambda'' + c^2\lambda^2 T_\lambda = 0 \quad \Rightarrow \quad T_\lambda = a_\lambda \cos(\lambda c t) + b_\lambda \sin(\lambda c t)$$

and the best way to ensure that $T_\lambda'(0) = 0$ is to take all $b_\lambda = 0$.

So now we have

$$\phi(r,\theta,t) = \sum_{\lambda,\alpha} c_\lambda R_{0,\lambda}(r)\cos(c\lambda t) \tag{13}$$

We also have the initial condition

$$f(r) = \phi(r,\theta,0) = \sum_\lambda c_\lambda R_{0,\lambda}(r) \tag{14}$$

Here we can use the orthogonality properties of the solutions $R_{0,\lambda}(r)$ (as developed in the preceding section) to uncover formulas for the coefficients c_λ. Multiplying (14) by $rR_{0,\lambda'}(r)$ and integrating between 0 and b yields

$$\begin{aligned}\int_0^b f(r)\,R_{0,\lambda'}(r)\,rdr &= \sum_{\lambda,} c_\lambda \int_0^b R_{0,\lambda}(r)\,R_{0,\lambda'}(r)\,rdr \\ &= \sum_{\lambda,} c_\lambda \left[\int_0^b R_{0,\lambda'}(r)\,R_{0,\lambda'}(r)\,rdr\right]\delta_{\lambda,\lambda'} \\ &= c_\lambda \int_0^b R_{0,\lambda'}(r)\,R_{0,\lambda'}(r)\,rdr\end{aligned}$$

Thus,

$$c_\lambda = \frac{1}{\int_0^b R_{0,\lambda'}(r)\,R_{0,\lambda'}(r)\,rdr}\int_0^b f(r)\,R_{0,\lambda'}(r)\,rdr \tag{15}$$

However, we are not yet done. We have yet to figure out exactly what λ we are summing over in (13) and, moreover, what exactly are the functions $R_{0,\lambda}(r)$.

5. Solution of the Radial Equation

First of all, we note that the radial equation (10) has a regular singular point at $r = 0$, and since we definitely want $r = 0$ to be within the domain of our solution, we are going to have to solve it using the generalized power series method.

However, before doing so, we can simplify things a little. The radial equation (for $\alpha = n = 0$) is

$$r^2 R_{0,\lambda}^{''} + rR_{0,\lambda}' + \lambda^2 r^2 R_{0,\lambda} = 0 \tag{16}$$

Let me introduce an auxiliary function $y(x)$ defined by

$$y(\lambda r) = R_{0,\lambda}(r) \tag{17}$$

We then have

$$\begin{aligned}\lambda y'(\lambda r) &= R_{0,\lambda}'(r) \\ \lambda^2 y''(\lambda r) &= R_{0,\lambda}''(r)\end{aligned}$$

and so y will satisfy (from (16))

$$r^2\lambda^2 y'' + r\lambda y'' + \lambda^2 r^2 y = 0$$

or, setting $x = \lambda r$:

$$x^2 y''(x) + xy(x) + x^2 y(x) = 0$$

or, even more simply,

$$xy'' + y' + xy = 0 \tag{18}$$

Equation (18) is the defining ODE for the a special function called the *Bessel function of order* 0.

Equation (17) has a regular singular point at $x = 0$, and so to solve it we'll have to resort to the Method of Frobenius. Thus, we set

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

and plug into to (17). We have

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} = r a_0 x^{r-1} + (r+1)\, a_1 x^r + \sum_{n=2}^{\infty} (n+r)\, a_n x^{n+r-1} \\ xy'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-1} = r\,(r-1)\, a_0 x^{r-1} + r\,(r+1)\, a_1 x^r + \sum_{n=2}^{\infty} a_n x^{n+r-1} \end{aligned}$$

Thus, to satisfy (17) we need

$$\begin{aligned} 0 &= [r\,(r-1)\, a_0 + r a_0]\, x^{r-1} + [r\,(r+1)\, a_1 + (r+1)\, a_1]\, x^r \\ &\quad + \sum_{n=2}^{\infty} [(n+r)\,(n+r-1)\, a_n + (n+r)\, a_n + a_{n-2}]\, x^{n+r-1} \end{aligned}$$

which leads to the following conditions

$$\begin{aligned} r^2 &= 0 \qquad \text{(the indicial equation)} \\ a_1 &= 0 \qquad \text{(from the vanishing of the toal coefficient of } x^r\text{)} \\ a_n &= \frac{-a_{n-2}}{(n+r)^2} \qquad \text{(the recursion relations)} \end{aligned}$$

Since we have only one root of the indicial equation, the discussion in Lecture 19, tells us that we'll only have one generalized power solution $y_1(x)$ corresponding to the root $r = 0$, and there'll be a second independent solution of the form

$$y_2(x) = y_1(x) \ln|x| + \sum_{n=1}^{\infty} b_n x^{n+0}$$

The second solution will thus have a logarthmic singularity at $x = 0$. Since $x = 0$ means the radial coordinate r is equal zero, and because we are only looking for solutions $R_{0,\lambda}(r)$ that make sense at $r = 0$, we are safe in ignoring the second solution.[1]

So it just remains to work out the solution corresponding to the one root of the indicial equation. Specializing the recursion relations to $r = 0$, we have

$$a_n = \frac{-a_{n-2}}{n^2}$$

The recurstion relations relate all the even coefficients to a_0 and all the odd coefficients will be proportional to a_1. Since $a_1 = 0$, we only have to worry about the even coefficients. We have

$$\begin{aligned} a_2 &= \frac{-a_0}{2^2} \\ a_4 &= \frac{-a_2}{4^2} = \frac{a_0}{2^2 4^2} = (-1)^{4/2+1} \frac{1}{2^2\,(1\cdot 2)^2} a_0 \\ a_6 &= \frac{-a_4}{6^2} = -\frac{a_0}{6^2 4^2 2^2} = (-1)^{6/2+1} \frac{1}{2^3\,(1\cdot 2\cdot 3)^2} \\ &\ \vdots \end{aligned}$$

and so it would seem

$$a_{2k} = \frac{(-1)^{k+1}}{2^k\,(k!)^2} a_0$$

Thus, our solution is

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)}{2^k\,(k!)^2} x^{2k}$$

[1]Sorry for the confusing notation where r can represent either the leading power of x in a generalized power series solution, or the radial coordinate. I hope the proper interpretation of r is always clear in its context.

It is not hard to see that the series on the right hand side actually converges for all x.

Finally, recall that

$$y(\lambda r) \equiv R_{0,\lambda}(r)$$

and so

$$R_{0,\lambda}(r) = \sum_{k=0}^{\infty} \frac{(-1)}{2^k (k!)^2} (\lambda r)^{2k} \tag{19}$$

will the form ouf our radial functions.

At this point, our ansatz for the solution of the vibrating drum problem has been made nearly explicit:

$$\phi(r,\theta,t) = \sum_{\lambda} c_\lambda R_{0,\lambda}(r)\cos(c\lambda t) \tag{20}$$

As the functions $R_{0,\lambda}(r)$ are given by (19) and the coefficients c_n are determined by (15);

$$c_\lambda = \frac{1}{\int_0^b R_{0,\lambda'}(r) R_{0,\lambda'}(r)\, r dr} \int_0^b f(r) R_{0,\lambda'}(r)\, r dr \qquad .$$

It remains to figure out the values of λ that occurring in the series on the right hand side of (20). These will be determined by requiring

$$R_{0,\lambda}(b) = 0.$$

6. Properties of Bessel Functions

A little more generally the solutions of the differential equation

$$x^2 y'' + xy' + x^2 y^2 - n^2 y = 0 \tag{21}$$

are called Bessel functions of order n. Let us denote the solution of (21) that is regular at $x = 0$ by $J_n(\xi)$; it turns out the Method of Frobenius produces solutions of the following form

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{n=0}^{\infty} \frac{(-1)}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k} \tag{22}$$

(note that this agrees with the particular case where $n = 0$ that we solved above).

6.1. Zeros of Bessel functions.

THEOREM 20.4. *All the zeros of $J_n(x)$ (other than $x = 0$) are simple; that is to say, if $x_0 \neq 0$ and $J_n(x_0) = 0$, then $J_n'(x_0) \neq 0$.*

Proof. (By contradiction). Suppose x_0 is a zero of $J_n(x)$ that is not simple. Evalating the differential for $J_n(x)$ at $x = x_0$ yields

$$\begin{aligned} 0 &= x_0^2 J_n''(x_0) + x_0 J_n'(x_0) + x_0^2 J_n(x_0) - n^2 J_n(x_0) \\ &= x_o^2 J_n''(x_0) + 0 + 0 + 0 \end{aligned}$$

So $J_n''(x_0) = 0$. But then by differentiating (21) further and evaluating it at x_0, one invitably finds that

$$\frac{d^k J_n}{dx^k}(x_0) = 0 \quad , \quad k = 0, 1, 2, 3, \ldots \tag{23}$$

Since $J_n(x)$ is an entire function (defined by a power series which converges everywhere), (23) would imply that $J_n(x) = 0$ for all x; which is a contradiction with its series expansion.

6.2. Asymptotic Expansion. It turns out that for sufficiently large x the functions $J_n(x)$ defined by (22) approach

$$J_n(x) \approx \sqrt{\frac{2}{nx}} \cos\left(x - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right)$$

since the graph of the right hand side oscillates above and below the x axis an infinite number of zeros as $x \to \infty$, so must the graph of $J_n(x)$. The Intermediate Value theorem then implies that $J_n(x)$ has an infinite number of zeros along the positive x axis.

In particular, this will be true for $J_0(x)$. Let's label and order the zeros of $J_0(x)$ as

$$0 = J_0(x_1) = J_0(x_2) = \cdots \qquad , \qquad 0 < x_1 < x_2 < x_3 < \ldots$$

If we now set

$$\lambda_i = \frac{x_i}{b} \tag{24}$$

and

$$R_{0,\lambda_i}(r) = J_0(\lambda_i r)$$

Then we will have

$$R_{0,\lambda_i}(b) = J_0\left(\frac{x_i}{b}b\right) = J_0(x_i) = 0$$

7. Conclusion

Let

$$0 < x_1 < x_1 < x_2 < \cdots$$

be the zeros of the Bessel function $J_0(x)$ and set

$$\lambda_i = \frac{x_i}{b}$$

Then

$$\phi(r,t) = \sum_{i=1}^{\infty} c_i J_0(\lambda_i r)\cos(c\lambda_i t)$$

will be the solution of our PDE/BVP, where

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

and the coefficients c_i are computable via

$$c_i = \frac{1}{\int_0^b J_{0,}(\lambda_i r) J_0(\lambda_i r)\, r dr} \int_0^b f(r) J_0(\lambda_i r)\, r dr$$