LECTURE 18

Power Series Solutions

1. Review of the Power Series Method

The Power Series method is established in Math 2233 as a general technique for solving a general second order, linear, homogeneous ODE of the form

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{1}$$

It provides, effectively, the Taylor expansion of a general solution that is valid in a neighborhood of any point x_0 for which the functions $p(x)$ and $q(x)$ are nice smooth functions. It goes like this, one suppose that there is a solution of the form of a power series about x_0

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{2}$$

and sets

$$p_m \equiv \frac{1}{m!} \frac{d^m p}{dx^m}(x_0)$$
$$q_m \equiv \frac{1}{m!} \frac{d^m q}{dx^m}(x_0)$$

so that $p(x)$ and $q(x)$ can also be expressed as power series about x_0

$$p(x) = \sum_{m=0}^{\infty} p_m (x - x_0)^m \tag{3}$$

$$q(x) = \sum_{m=0}^{\infty} q_m (x - x_0)^m \tag{4}$$

One has

$$y'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \tag{5}$$

$$y^{\prime\prime}(x) = \sum_{n=0}^{\infty} n(n-1)\, a_n (x - x_0)^{n-2} \tag{6}$$

Replacing $y, y', y'', p(x)$ and $q(x)$ in (1) by their power series expressions (2) – (6), one obtains

$$0 = \sum_{n=0}^{\infty} n(n-1)\, a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n a_n p_m (x - x_0)^{m+n-1} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n q_m (x - x_0)^{m+n} \tag{7}$$

One then uses power series manipulations (shifts of summation indices, peeling off initial terms, etc) so that the right hand side of (7) can be written as single power series

$$0 = \sum_{n=0}^{\infty} F_n(n, a_*, p_*, q_*)\,(x - x_0)^n \tag{8}$$

Here $F_n(n, a_*, p_*, q_*)$ is the total coefficient of $(x - x_0)^n$ on the right side of (7), it will depend n and some of the coefficients $a_0, a_1, \dots, a_n, p_0, \dots, p_n, q_0, \dots, q_n$. Using the principle that a power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n$$

equals zero only if every coefficient A_n separately equals 0, we get from (8) and infinite set of equations:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 0 \\ F_2 &= 0 \\ &\vdots \end{aligned} \tag{9}$$

and these (plus our knowledge of $p_0, p_1, p_2, \dots$ and $q_0, q_1, q_2, \dots$) allow one to compute all the coefficients $a_2, a_3, a_4, \dots$ in terms of the first two a_0 and a_1. For a general solution of (1), the constants a_0 and a_1, left undetermined by the equations (9) are left as pair of arbitrary constants. On the other hand, if initital conditions are given at x_0

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned}$$

then the constants a_0 and a_1 correspond precisely to these initial values:

$$\begin{aligned} a_0 &= y(x_0) \\ a_1 &= y'(x_0) \end{aligned}$$

(as one should expect; since a power series solution is the same thing as the Taylor expansion of a solution and the first two coefficients of a Taylor expansion about x_0 are $y(x_0)$ and $y'(x_0)$).

Let me do a simple example to help you better recall the power series technique.

EXAMPLE 18.1. Solve the following initial value problem using the Power Series method:

$$\begin{aligned} y'' - x^2 y &= 0 \\ y(1) &= 2 \\ y'(1) &= 4 \end{aligned}$$

- Since initial conditions are defined at $x = 1$, we will have to use a power series about $x = 1$. But at least we know

  $$a_0 = 2 \qquad \text{and} \qquad a_1 = 5$$

  Substituting $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ and

  $$x^2 = 1 + 2(x-1) + (x-1)^2 \qquad \text{(the Taylor expansion of } f(x) = x^2 \text{ about } x = 1\text{)}$$

  into the differential equation yields

  $$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} - \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n \\ &- \sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=1}^{\infty} 2a_{n-1} (x-1)^n - \sum_{n=2}^{\infty} a_{n-2} (x-1)^n \end{aligned}$$

$$
\begin{aligned}
&= 0 + 0 + 2a_2 + +3a_3 (x-1) + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n \\
&- a_0 - a_1 (x-1) + \sum_{n=2}^{\infty} -a_n (x-1)^n \\
&- 2a_0 (x-1) + \sum_{n=2}^{\infty} -2a_n (x-1)^n \\
&+ \sum_{n==2}^{\infty} a_{n-2} (x-1)^n
\end{aligned}
$$

or

$$0 = (2a_2 - a_0) + (3a_3 - a_1 - 2a_0)(x-1) + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - a_n - 2a_{n-1} - a_{n-2}] (x-1)^n$$

We thus require

$$
\begin{aligned}
0 = 2a_2 - a_0 \quad &\Rightarrow \quad a_2 = \frac{1}{2} a_0 \\
0 = 3a_3 - a_1 - 2a_0 \quad &\Rightarrow \quad a_3 = \frac{1}{3}(2a_0 + a_1)
\end{aligned}
$$

and

$$
\begin{aligned}
&0 = (n+2)(n+1) a_{n+2} - a_n - a_{n-1} - a_{n-2} \qquad , \qquad n = 2, 3, 4, \ldots \\
&\Rightarrow \quad a_{n+2} = \frac{a_n + 2a_{n-1} + a_{n-2}}{(n+2)(n+1)}
\end{aligned}
$$

Now using the known values for a_0 and a_1 (coming from our initial conditions)

$$
\begin{aligned}
a_0 &= 2 \\
a_1 &= 5 \\
a_2 &= 1 \\
a_3 &= \frac{1}{3}(4+5) = 3 \\
a_4 &= a_{2+2} = \frac{a_2 + 2a_1 + a_0}{(4)(5)} = \frac{1 + 10 + 2}{20} = \frac{13}{20} \\
&\vdots
\end{aligned}
$$

and so our solution to order $(x-1)^4$ is

$$y(x) = 2 + 5(x-1) + (x-1)^2 + 3(x-1)^3 + \frac{13}{20}(x-1)^4 + \cdots$$

2. Limitations of the Power Series Method

Implicit in the technique given above is the requirement that the coefficient functions $p(x)$ and $q(x)$ in the original differential equations had Taylor series expansions about the expansion point x_0. Sometimes, however, it will be precisely at a point where the functions $p(x)$ or $q(x)$ are singular where we are most interested in understanding the solution. This, for example, will happen latter when we try to use solutions of Bessel's equation to develop a Separation of Variables type solution of a vibrating drum head problem. The drum head's motion at its center will correpond to a point where differential equation for the radial component has a term $q(x)$ that goes like $\frac{1}{r}$. We will discuss a "Generalized Power Series Method" in the next lecture to deal with the problems of constructing solutions of "singular" differential that are valid right up to (and sometimes including) the singularities of the coefficient fuctions $p(x)$ and $q(x)$.

For now, however, we'll focus on another limitation on a power series solution: **power series do not always define valid functions**.

You see in order to evaluate a power series at a point x, one can not simply sum an infiinite set of numbers (as there is in fact no way to complete this infinite operation). Rather one has to take the limit of the sequence of partial sums: if

$$f(x) \equiv \sum_{n=0}^{\infty} a_n x^n \tag{11}$$

then

$$f(3) \equiv \lim_{N \to \infty} \sum_{n=0}^{N} a_n 3^n$$

and the problem is that the limit on the right might not exist.

The conditions under which a power series like (11) converge is usually covered in Calculus II (as a precursor to defining the Riemann integral). In that setting there is a large menagerie of tests (the integral test, the ratio test, the root test, etc) on the coefficients for checking for when a given numerical series makes converges, and through these tests a means for checking when a power series makes sense as a function.

Luckily we do not need all that apparatus to figure out for what values of x a power series solution actually provides a function of x. We just need a couple of definitions and a big theorem.

DEFINITION 18.2. *The* ***radius of convergence*** *of a power series*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is the smallest value of $|x - x_0|$ *for which*

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n \qquad \textit{exists.}$$

THEOREM 18.3. *If* R *is the radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges to a legitimate (in fact, analytic) function for all x *such that* $|x - x_0| < R$*, and it diverges as a series for all* x *such that* $|x - x_0| > R$.

DEFINITION 18.4. *A function* f *is said to be* ***analytic*** *at* x_0 *if it has a Taylor expansion about* x_0 *and this Taylor expansion converges for all points in an* ε*-neighborhood of* x_0 :

$$\lim_{N \to \infty} \sum_{n=0}^{\infty} \frac{d^n f}{dx^n}(x_0)(x - x_0)^n \ \ \textit{exists for all } x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$

If f *is analytic at* x_0*, we say* x_0 *is a* ***regular point*** *for* f*. Contrarily, if* $f(x)$ *is not analytic at* x_0*, but at every other point* x *in an* ε*-neighborhood of* x_0 *it is analytic, we say that* x_0 *is a* ***singular point*** *for* f*.*

The regular and singular points of function prescribed by a formula are usually easy to identify:

EXAMPLE 18.5. Consider the function

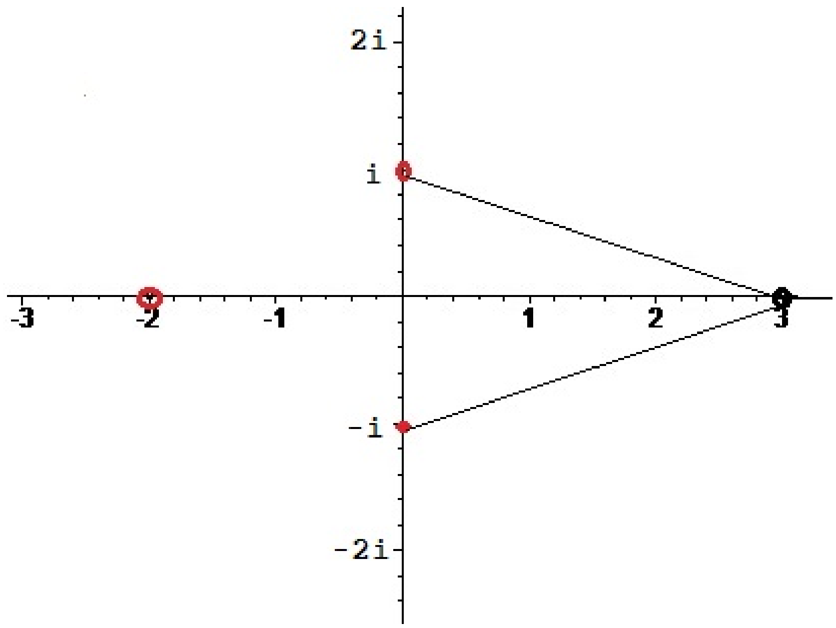
$$f(x) = \frac{2x + 1}{(x - 1)(x + 2)}$$

The singular points are the points where one factor of the other of the denominator vanishes. Thus,

$$\text{singular points of } f = \{1, -2\}$$

Every other point is regular

$$\text{regular points of } f = \mathbb{R} - \{1, 2\}$$



THEOREM 18.6. *Suppose the functions $p(x)$ and $q(x)$ are analytic at x_0 and*

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{13}$$

is a formal power series solution to

$$y'' + p(x) y' + q(x) y = 0$$

Then the formal power series (13) converges to a bona fide function of x for all x such that $|x - x_0|$ is less than the distance from x_0 to the closest singular point of p and q in the complex plane.

EXAMPLE 18.7. Consider the differential equation

$$\left(x^2 + 1\right) y'' + y' + \frac{x^2 + 1}{x + 2} y = 0$$

What is the minimal radius of convergence for a power solution about $x = 3$.

- To answer this question we need to figure out the singularities of

$$p(x) = \frac{1}{(x^2 + 1)} \qquad \text{and} \qquad q(x) = \frac{1}{(x + 2)}$$

Evidently, one or the other of these functions are singular at $x = \pm i$ and $x = -2$. Let's plot these points in the complex plane and then compute their distances from the expansion point $x = 3$. We have (using plane geometry to compute distances)

$$\|3 - i\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$$

$$\|3 - (-i)\| = \sqrt{(3)^2 + (-1)^2 = \sqrt{10}}$$

$$\|3 - (-2)\| = \sqrt{(5)^2 + (0)^2 = 5}$$

Since the shortest distance from a singular point to the expansion point $x = 3$ is $\sqrt{10}$, a power series solution about $x = 3$ will converge for all x such that

$$|x - 3| < \sqrt{10} \quad \Rightarrow \quad x \in \left(3 - \sqrt{10}, 3 + \sqrt{10}\right)$$

Note that we could reach this conclusion without even calculating the power sereis solution.

In summary, there is a straight-forward procedure for solving differential equations of the form

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Moreover, it is relatively easy to determined the domain of validity of the resulting power series solution. The hedge though, is this procedure doesn't give us any information at all about the nature of solutions close to a singularity. Determining solutions near a singular point is the topic of the next lecture.