Lecture 10

Laplace’s Equation

So far we’ve discussed the heat equation

\[ \frac{\partial T}{\partial t} - \alpha^2 \nabla^2 T = 0, \]

and the wave equation

\[ \frac{\partial^2 \phi}{\partial t^2} - \alpha^2 \nabla^2 \phi = 0. \]

The last prototypical PDE is Laplace’s equation, which is

\[ \nabla^2 \phi = 0. \]

Laplace’s equation arises in a number of physical applications, one actually follows immediately from our discussion of the heat equation. Consider a system governed by the heat equation that is allowed to reach a time-independent state of equilibrium. In its equilibrium state we’ll have

\[ T(x, t) = T_{ss}(x) \]

which will obey

\[ 0 = \frac{\partial T}{\partial t} - \alpha^2 \nabla^2 T = 0 - \alpha^2 \nabla^2 T_{ss} \quad \implies \quad \nabla^2 T_{ss} = 0 \]

1. Separation of Variables

In the following we’ll consider the 2-dimensional Laplace equation

\[ 0 = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \]

and look for solutions of the form

\[ \phi(x, y) = X(x)Y(y). \]

Plugging (5) into (4) and then dividing both sides by \( X(x)Y(y) \) yields

\[ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \]

Applying the by now familiar separation-of-variables argument, we conclude that \( X(x) \) and \( Y(y) \) must satisfy equations of the form

(6a) \[ X''(x) = CX(x) \]

(6b) \[ Y''(y) = -CY(y) \]
2. Dirichlet Boundary Conditions

To make further progress towards a solution we’ll now restrict attention to a particular physical situation with a particular set of boundary conditions. Consider a rectangular plate, three sides of which are immersed in a heat bath so that their temperatures are maintained at 0, and one side of which has its temperature maintained at prescribed function of $y$:

\[ T(0,y) = 0 \]

\[ T(a,y) = f(y) \]

\[ T(x,0) = 0 \]

The boundary conditions require

\[ 0 = T(x,0) = X(x)Y(0) \quad \Rightarrow \quad Y(0) = 0 \]  

\[ 0 = T(x,b) = X(x)Y(b) \quad \Rightarrow \quad Y(b) = 0 \]  

\[ 0 = T(0,y) = X(0)Y(y) \quad \Rightarrow \quad X(0) = 0 \]  

\[ f(y) = T(a,y) \]

The boundary conditions on the right of (7a) and (7b) together with the differential equation (6b) require

\[ C = \frac{n\pi}{b}, \quad n = 1, 2, \ldots \]

and

\[ Y(y) = \sin \left( \frac{n\pi}{b} y \right) \]

by an argument we have worked out several times before.

With $C = \frac{n\pi}{b}$ the general solution of (6a) will be

\[ X(x) = c_1 \cosh \left( \frac{n\pi}{b} x \right) + c_2 \sinh \left( \frac{n\pi}{b} x \right) \]

which will satisfy the boundary condition (7c) only if we take $c_2 = 0$. If we now set

\[ T(x,y) = \sum_{n=1}^{\infty} c_n \sinh \left( \frac{n\pi}{b} x \right) \sin \left( \frac{n\pi}{b} y \right) \]

then we have a solution not only of the PDE but also three out of the four boundary conditions. It remains to adjust the coefficients $c_n$ so that the last boundary condition is satisfied

\[ f(y) = T(a,y) \quad \Rightarrow \quad f(y) = \sum_{n=1}^{\infty} c_n \sinh \left( \frac{n\pi}{b} a \right) \sin \left( \frac{n\pi}{b} y \right) \]

Employing the Fourier-sine expansion of $f(y)$

\[ f(y) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{b} y \right) \quad \text{with} \quad b_n = \frac{2}{b} \int_{0}^{b} f(x) \sin \left( \frac{n\pi}{b} y \right) dy \]
we can conclude that the solution of the PDE and boundary conditions is given by
\[
T(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad \text{with} \quad c_n = \frac{2}{b \sin\left(\frac{n\pi a}{b}\right)} \int_0^b f(x) \sinh\left(\frac{n\pi}{b}y\right) \, dy
\]