LECTURE 1

Linear Algebra and Matrices

Before embarking on a study of systems of differential equations we will first review, very quickly, some fundamental objects and operations in linear algebra.

1. Matrices

DEFINITION 1.1. *An* $\mathbf{n} \times \mathbf{m}$ ***matrix*** *("n by m matrix") is an arrangement of nm objects (usually numbers) into a rectangular array with n rows and m columns. We'll typically denote the entry in the* i^{th} *row and* j^{th} *column of a matrix* $\mathbf{A}$ *as* a_{ij} *(and similarly,* b_{ij} *for the* $(ij)^{th}$ *entry of a matrix* $\mathbf{B}$*). Thus, for example,*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

An $n \times 1$ *matrix*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is called a ***column vector*** *and a* $1 \times m$ *matrix*

$$\mathbf{v} = [v_1, v_2, \ldots, v_m]$$

is called a ***row vector****.*

1.1. Matrix Operations.

DEFINITION 1.2. *The* ***sum*** *of two* $n \times m$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *is the* $n \times m$ *matrix* $\mathbf{A} + \mathbf{B}$ *with entries*

$$(\mathbf{A} + \mathbf{B})_{ij} = a_{ij} + b_{ij} \quad .$$

DEFINITION 1.3. *The* ***scalar product*** *of an* $n \times m$ *matrix* $\mathbf{A}$ *with a number* λ *is the* $n \times m$ *matrix* $\lambda\mathbf{A}$ *with entries*

$$(\lambda\mathbf{A})_{ij} = \lambda a_{ij} \quad .$$

DEFINITION 1.4. *The* ***difference*** *of two* $n \times m$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *is the* $n \times m$ *matrix* $\mathbf{A} - \mathbf{B}$ *with entries*

$$(\mathbf{A} - \mathbf{B})_{ij} = a_{ij} - b_{ij} = (\mathbf{A} + (-1)\,\mathbf{B})_{ij}$$

DEFINITION 1.5. *The* ***transpose*** *of an* $n \times m$ *matrix* $\mathbf{A}$ *is the* $m \times n$ *matrix* $\mathbf{A}^t$ *with entries*

$$\left(\mathbf{A}^t\right)_{ij} = A_{ji}$$

(Note the i^{th} *row of* $\mathbf{A}^t$ *is simply the* j^{th} *column of* $\mathbf{A}$ *written horizontally.)*

DEFINITION 1.6. *An* $n \times n$ *matrix* $\mathbf{A}$ *is said to be* ***symmetric*** *if*

$$\mathbf{A} = \mathbf{A}^t \quad .$$

Note that the transpose of a row-vector is a column vector and the transpose of a column vector is a row vector.

DEFINITION 1.7. *The **dot product** of an n-dimensional row vector $[r_1, \ldots, r_n]$ with a n-dimensional column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the number*

$$[r_1, \ldots, r_n] \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r_1 c_1 + \cdots + r_n c_n$$

(This coincides with the usual dot products of vectors when we think of vectors as ordered lists of numbers rather than special kinds of matrices.)

DEFINITION 1.8. *The **matrix product** of an $n \times m$ matrix $\mathbf{A}$ with a $m \times q$ matrix $\mathbf{B}$ is the $n \times q$ matrix with entries*

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Note that the $(ij)^{th}$ entry of the matrix product $\mathbf{AB}$ is the dot product of the i^{th} row of $\mathbf{A}$ with the j^{th} column of $\mathbf{B}$.

2. Complex Numbers

In what follows it will be sometime necessary to consider vectors and matrices with entries that are complex numbers. We recall here some basic facts about complex numbers.

DEFINITION 1.9. *A complex numbers are pairs (x, y) of real numbers which satisfying the following addition and multiplication rules*

$$\begin{aligned} (x, y) + (x', y') &= (x + x', y + y') \\ (x, y) \cdot (x', y') &= (xx' - yy', xy' + yx') \end{aligned}$$

Usually though we write a complex number as

$$z = x + iy$$

and then add and multiply complex numbers by employing the usual rules of arithmetic and the relation $i^2 = -1$. Thus,

$$\begin{aligned} (x + iy)(x' + iy') &= xx' + x(iy') + iy(x') + (iy)(iy') \\ &= xx' - yy' + i(xy' + yx') \end{aligned}$$

When we write

$$z = x + iy$$

we say that x is the **real** part of z and y is the **imaginary** part of z.

The **complex conjugate** of $z = x + iy$ is the complex number $\overline{z}$ obtained by switching the sign of it imaginary part

$$\overline{z} = x - iy$$

We have

$$\begin{aligned} Re(z) &= x = \frac{z + \overline{z}}{2} \\ Im(z) &= y = \frac{z - \overline{z}}{2i} \\ &Re() \end{aligned}$$

We will also sometimes consider functions of a complex variable. Here we only state the very fundamental Euler Formula

$$e^{x+iy} = e^x \left(\cos(y) + i \sin(y)\right)$$

We note that

$$\begin{aligned} \cos(y) &= Re\left(e^{iy}\right) \\ \sin(y) &= Im\left(e^{iy}\right) \end{aligned}$$

DEFINITION 1.10. *A **complex matrix** is a matrix whose entries are complex numbers. Similarly, a **complex vector** is a vector whose components are complex numbers. The **hermitian adjoint** of a complex matrix* $\mathbf{A}$ *is the matrix* $\mathbf{A}^\dagger$ *with entries*

$$\left(\mathbf{A}^\dagger\right)_{ij} = \overline{a}_{ji}$$

Equivalently,

$$\mathbf{A}^\dagger = \left(\overline{\mathbf{A}}\right)^t = \overline{\left(\mathbf{A}^t\right)}$$

EXAMPLE 1.11.

$$\mathbf{A} = \begin{pmatrix} 1+i & 1-2i \\ 3+i & 3 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{A}^\dagger = \begin{pmatrix} 1-i & 3-i \\ 1+2i & 3 \end{pmatrix}$$

DEFINITION 1.12. *An* $n \times n$ *matrix is **hermitian** (or **self-adjoint**) if*

$$\mathbf{A} = \mathbf{A}^\dagger$$

DEFINITION 1.13. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two complex (column) vectors. The (**hermitian**) inner product* $(\mathbf{x}, \mathbf{y})$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is the complex number*

$$\mathbf{x}^t \overline{\mathbf{y}} = [x_1, \cdots, x_n] \begin{bmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{bmatrix} = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$$

REMARK 1.14. If $\mathbf{A}$ is a real hermitian matrix (that is, all of its entries are real numbers), then

$$\mathbf{A}^\dagger = \left(\overline{\mathbf{A}}\right)^t = (\mathbf{A})^t = \mathbf{A}^t$$

so a real hemitian matrix is just a real symmetric matrix. If $\mathbf{x}$ and $\mathbf{y}$ are two real vectors then the hermitian inner product $(\mathbf{x}, \mathbf{y})$ coincides with the usual dot product of real vectors.

3. Determinants

DEFINITION 1.15. *The* $(ij)^{th}$ ***minor** of an* $n \times n$ *matrix* $\mathbf{A}$ *is the* $(n-1) \times (n-1)$ *matrix* $\mathbf{M}_{ij}$ *obtained by deleting the* i^{th} *row and* j^{th} *column from* $\mathbf{A}$.

DEFINITION 1.16. *The determinant of an* $n \times n$ *matrix* $\mathbf{A}$ *is the number* $\det \mathbf{A}$ *determined by the following recursive algorithm:*

- *If* $\mathbf{A}$ *is a* 1×1 *matrix* $[a_{11}]$, *then* $\det \mathbf{A} = a_{11}$
- if $\mathbf{A}$ is an $n \times n$ matrix then

$$\begin{aligned} \det \mathbf{A} \quad &: = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij} \qquad (i = \text{any fixed row index}) \\ &: = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij} \qquad (j = \text{any fixed column index}) \end{aligned}$$

The following formulas for the determinants of 2×2 and 3×3 matrices follow easily from the above definition

$$\begin{aligned}\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} &= (-1)^{1+1} a_{11}\det(M_{11}) + (-1)^{1+2} a_{12}\det(M_{12}) \\ &= a_{11}\det[a_{22}] - a_{12}\det[a_{21}] \\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

$$\begin{aligned}\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} &= (-1)^{1+1} a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{bmatrix} + (-1)^{1+2} a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{bmatrix} \\ &\quad + (-1)^{1+3} a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})\end{aligned}$$

4. Inverses

DEFINITION 1.17. *The **inverse** of an $n\times n$ matrix $\mathbf{A}$ is the $n\times n$ matrix $\mathbf{A}^{-1}$ such that*

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

(N.B. matrix inverses do not always exist.)

There are two basic ways of computing matrix inverses:

- Row reduction:
$$[\mathbf{A} \mid \mathbf{I}] \quad \xrightarrow{\textit{row reduction}} \quad [\mathbf{I} \mid \mathbf{A}^{-1}]$$
- Cofactor Method:
$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^t$$
where $\mathbf{C}$ is the cofactor matrix of $\mathbf{A}$ whose entries are ± 1 times the determinants of the $(n-1)\times(n-1)$ minors of $\mathbf{A}$.
$$(\mathbf{C})_{ij} = (-1)^{i+j}\det(\mathbf{M}_{ij})$$

5. Eigenvalues and Eigenvectors

DEFINITION 1.18. *Let $\mathbf{A}$ be an $n\times n$ matrix. If there exists a number λ and an n-dimensional (column) vector $\mathbf{v}$ such that*

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

*then $\mathbf{v}$ is said to be an **eigenvector** of $\mathbf{A}$ and λ is said to be the **eigenvalue** of $\mathbf{A}$ corresponding to the eigenvector $\mathbf{v}$.*

The following algorithm determines all the eigenvectors and eigenvalues of an $n\times n$ matrix $\mathbf{A}$.

- Set $p_{\mathbf{A}} = \det(\mathbf{A} - \lambda\mathbf{I})$. Here λ is regarded as a variable, and $\lambda\mathbf{I}$ is the matrix obtained by scalar multiplying the $n\times n$ identity matrix by λ.

  $\mathbf{A} - \lambda\mathbf{I}$ is thus the matrix obtained by subtracting λ from each of the diagonal entries of $\mathbf{A}$. Upon computation, $\det(\mathbf{A} - \lambda\mathbf{I})$ will yield a polynomial of degree n in λ. The solutions of
$$p_{\mathbf{A}}(\lambda) = 0$$
will be the eigenvalues of $\mathbf{A}$.

- For each root $\lambda = r$ of $p_{\mathbf{A}}(\lambda) = 0$, find the general solution of
$$(\mathbf{A} - r\mathbf{I})\,\mathbf{x} = \mathbf{0}$$
and expand that solution in terms of the free parameters of the solution. The eigenvectors corresponding to the eigenvalue r of $\mathbf{A}$ will be the constant vectors in that expansion.

EXAMPLE 1.19. Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

- We have
$$p_{\mathbf{A}}(\lambda) := \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$
Since $\lambda = 3, -1$ are the solutions of $p_{\mathbf{A}}(\lambda) = 0$, these are the eigenvalues of $\mathbf{A}$.
- Now we look for the eigenvectors corresponding to $\lambda = 3$.
$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longleftrightarrow\quad \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
The reduced row echelon form of the coefficient matrix is
$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \quad\rightarrow\quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad\Longrightarrow\quad \left\{ \begin{aligned} x_1 - x_2 &= 0 \\ 0 &= 0 \end{aligned} \right. \quad\Longrightarrow\quad x_1 = x_2$$
$$\Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
so
$$\mathbf{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
- Similarly, we look for the eigenvectors corresponding to $\lambda = -1$:
$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longleftrightarrow\quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longrightarrow\quad x_1 = -x_2$$
$$\Longrightarrow \quad \mathbf{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
and so
$$\mathbf{v}_{\lambda=-1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

EXAMPLE 1.20. Suppose
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
Find the eigenvectors and eigenvalues of $\mathbf{A}$.

- The characteristic polynomial is
$$p_{\mathbf{A}}(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$$
To solve $p_{\mathbf{A}}(\lambda) = 0$, we apply the quadratic formula
$$ax^2 + bx + c = 0 \quad\Longrightarrow\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
and find
$$p_{\mathbf{A}}(\lambda) = 0 \quad\Longrightarrow\quad \lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{-1}\sqrt{4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$
Thus we have two *complex* eigenvalues.

- Setting $\lambda = 1 + i$, we now solve
$$\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
The row reduction method of solving linear systems still works:
$$\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} \xrightarrow{R_2 \to R_2 + iR_1} \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to (-i)\, R_1} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$
$$\Longrightarrow \quad x_1 = -ix_2 \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}_{\lambda=1+i} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$
- Similarly, for $\lambda = 1 - i$, one finds
$$\mathbf{v}_{\lambda=1-i} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

6. Diagonalization of Matrices

Recall that a **diagonal matrix** is a square $n \times n$ matrix with non-zero entries only along the diagonal from the under left to the lower right (the *main diagonal*).

Diagonal matrices are particularly convenient for eigenvalue problems since the eigenvalues of a diagonal matrix
$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$
coincide with the diagonal entries $\{a_{ii}\}$ and the eigenvector corresponding the eigenvalue a_{ii} is just the i^{th} coordinate vector. That is, if $\mathbf{A}$ is of the above form, we always have
$$\mathbf{A}\mathbf{e}_i = a_{ii}\mathbf{e}_i$$

DEFINITION 1.21. *An $n \times n$ matrix $\mathbf{A}$ is **diagonalizable** if there is an invertible $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C}$ is a diagonal matrix. The matrix $\mathbf{C}$ is said to **diagonalize** $\mathbf{A}$.*

LEMMA 1.22. *Let $\mathbf{A}$ be a real (or complex) $n \times n$ matrix, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a set of n real (respectively, complex) scalars, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a set of n vectors in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$). Let $\mathbf{C}$ be the $n \times n$ matrix formed by using $\mathbf{v}_j$ for j^{th} column vector, and let $\mathbf{D}$ be the $n \times n$ diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*
$$\mathbf{AC} = \mathbf{CD}$$
if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$ and each $\mathbf{v}_j$ is an eigenvector of $\mathbf{A}$ correponding the eigenvalue λ_j .

Now suppose $\mathbf{AC} = \mathbf{CD}$, and the matrix $\mathbf{C}$ is invertible. Then we can write
$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC}.$$
And so we can think of the matrix $\mathbf{C}$ as converting $\mathbf{A}$ into a diagonal matrix.

THEOREM 1.23. *An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors.*

EXAMPLE 1.24. Find the matrix that diagonalizes
$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$$

- First we'll find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) \quad \Rightarrow \quad \lambda = 2, -1$$

The eigenvectors corresponding to the eigenvalue $\lambda = 2$ are solutions of $(\mathbf{A} - (2)\mathbf{I})\,\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} 6x_2 = 0 \\ -3x_2 = 0 \end{array} \quad \Rightarrow \quad x_2 = 0 \quad \Rightarrow \quad \mathbf{x} = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenvectors corresponding to the eigenvalue $\lambda = -1$ are solutions of $(\mathbf{A} - (-1)\mathbf{I})\,\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} 3x_1 + 6x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad x_1 = -2x_2 \quad \Rightarrow \quad \mathbf{x} = r \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So the vectors $\mathbf{v}_1 = [1, 0]$ and $\mathbf{v}_2 = [-2, 1]$ will be eigenvectors of $\mathbf{A}$. We now arrange these two vectors as the column vectors of the matrix $\mathbf{C}$.

$$\mathbf{C} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

In order to compute the diagonalization of $\mathbf{A}$ we also need $\mathbf{C}^{-1}$. This we compute using the technique of Section 1.5:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right] \quad \Rightarrow \quad \mathbf{C}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Finally,

$$\begin{aligned} \mathbf{D} &= \mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{C}^{-1}(\mathbf{A}\mathbf{C}) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$