1. Determine if the following ODEs are of the Sturm-Liouville type
\[
\frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] - q(x) y + \lambda r(x) y = 0, \quad p(x) > 0, \quad r(x) > 0
\]
and if so identify the functions \( p(x), q(x) \) and \( r(x) \).

(a) \((1 + x^2) y'' - 2xy' + l(l+1) y = 0\)

(b) \(x^2y'' + xy' + (x^2 - n^2) y = 0\)

2. Develop a formal eigenfunction expansion for the solution of the following problem
\[
y'' + \mu y = -f(x), \quad y'(0) = 0, \quad y(1) + y'(1) = 0
\]

3. Let \( L \) be a second order linear differential operator. Show that the solution \( y = \phi(x) \) of the problem
\[
L[y] = f(x), \quad \alpha_1 y(0) + \alpha_2 y'(0) = a, \quad \beta_1 y(1) + \beta_2 y'(1) = b
\]
can be written
\[
y(x) = u(x) + v(x)
\]
where \( u(x) \) and \( v(x) \) are solutions to the problems
\[
L[u] = 0, \quad \alpha_1 u(0) + \alpha_2 u'(0) = a, \quad \beta_1 u(1) + \beta_2 u'(1) = b
\]
and
\[
L[v] = f(x), \quad \alpha_1 v(0) + \alpha_2 v'(0) = 0, \quad \beta_1 v(1) + \beta_2 v'(1) = 0
\]

3. Show by the method of Variation of Parameters that the general solution of
\[
y'' = -f(x)
\]
can be written
\[
y(x) = c_1 + c_2 x - \int (x-x) f(s) \, ds
\]

4. Solve
\[
y'' = -f(x), \quad y'(0) = 0, \quad y(1) = 0
\]
by determining the appropriate Green’s function and expressing the solution as a definite integral.

5. Express the 2-dimensional Helmholtz equation
\[
\nabla^2 \phi = -k^2 \phi
\]
in polar coordinates and then use Separation of Variables and Sturm-Liouville theory to find the solution that remains bounded at all points on the disk \(0 \leq r \leq 1\), that is periodic with period \(2\pi\), and satisfies the boundary condition
\[
\phi(1, \theta) = f(\theta)
\]
where \( f(\theta) \) is a continuous function on the interval \([0, 2\pi]\).