1. Let $\mathcal{P}$ be the vector space of polynomials with indeterminant $x$. Which of the following mappings are linear transformations from $\mathcal{P}$ to itself?

(a) $T : p \to xp$

- Let $p_1, p_2$ be two polynomials and let $\alpha, \beta \in \mathbb{F}$. We have
  
  \begin{align*}
  p_1 &= a_n x^n + \cdots + a_1 x + a_0 \\
  p_2 &= b_m x^m + \cdots + b_1 x + b_0
  \end{align*}

  If $n \neq m$ we can anyway replace the polynomial of lower degree (e.g. say it’s $p_2$) with its equivalent expression
  
  $$p_2 = 0 \cdot x^n + 0 \cdot x^{n-1} + \cdots + 0 \cdot x^{m+1} + b_m x^m + \cdots + b_1 x + b_0$$

  So we can without loss of generality write
  
  \begin{align*}
  p_1 &= a_n x^n + \cdots + a_1 x + a_0 \\
  p_2 &= b_n x^n + \cdots + b_1 x + b_0
  \end{align*}

  for a pair of arbitrary polynomials. Then by the definition of scalar multiplication and addition in $\mathcal{P}$ we’ll have
  
  $$\alpha p_1 + \beta p_2 = (\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)$$

  and so
  
  \begin{align*}
  T(\alpha p_1 + \beta p_2) &= x [(\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)] \\
  &= (\alpha a_n + \beta b_n) x^{n+1} + \cdots + (\alpha a_1 + \beta b_1) x^2 + (\alpha a_0 + \beta b_0) x \\
  &= (\alpha a_n) x^{n+1} + \cdots + (\alpha a_1) x^2 + (\alpha a_0) x \\
  &\quad + (\beta b_n) x^{n+1} + \cdots + (\beta b_1) x^2 + (\beta b_0) x \\
  &= \alpha x (a_n x^n + a_1 x + a_0) + \beta x (b_n x^n + \cdots + b_1 x + b_0) \\
  &= \alpha T(p_1) + \beta T(p_2)
  \end{align*}

  Since $T$ preserves arbitrary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. \(\square\)

(b) $T : p \to 2p$

- Using the same setup as in preceding problem, we compute
  
  \begin{align*}
  T(\alpha p_1 + \beta p_2) &= 2 [(\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)] \\
  &= (2\alpha a_n + 2\beta b_n) x^n + \cdots + (2\alpha a_1 + 2\beta b_1) x + (2\alpha a_0 + 2\beta b_0) \\
  &= \alpha 2 (a_n x^n + a_1 x + a_0) + \beta 2 (b_n x^n + \cdots + b_1 x + b_0) \\
  &= \alpha T(p_1) + \beta T(p_2)
  \end{align*}

  Since $T$ preserves arbitrary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. \(\square\)

(c) $T : p \to \frac{dp}{dx} + 2p$
Here we won’t be so explicity as in parts (a) and (b); we’ll simply use the facts that differentiation operates term by term and commutes with scalar multiplication.

\[ T(\alpha p_1 + \beta p_2) = \left( \frac{d}{dx} + 2 \right) (\alpha p_1 + \beta p_2) \]
\[ = \alpha \frac{dp_1}{dx} + 2\alpha p_1 + \beta \frac{dp_2}{dx} + 2\beta p_2 \]
\[ = \alpha \left( \frac{d}{dx} + 2 \right) p_1 + \beta \left( \frac{d}{dx} + 2 \right) p_2 \]
\[ = \alpha T(p_1) + \beta T(p_2) \]

Since \( T \) preserves arbitrary linear combinations of elements of \( P \), it is a linear transformation. \( \square \)

(d) \( T : p \to \int_0^1 p(x) \, dx \)

Here we’ll just use the facts that we can integrate term by term and pull constants through integral signs.

\[ T(\alpha p_1 + \beta p_2) = \int_0^1 (\alpha p_1 + \beta p_2)(x) \, dx \]
\[ = \int_0^1 (\alpha p_1(x) + \beta p_2(x)) \, dx \]
\[ = \int_0^1 \alpha p_1(x) \, dx + \int_0^1 \beta p_2(x) \, dx \]
\[ = \alpha \int_0^1 p_1(x) \, dx + \beta \int_0^1 p_2(x) \, dx \]
\[ = \alpha T(p_1) + \beta T(p_2) \]

Since \( T \) preserves arbitrary linear combinations of elements of \( P \), it is a linear transformation. \( \square \)

2. Suppose \( f : V \to W \) is a linear transformation.

(a) Prove that \( f \) is injective if and only if \( \ker(f) = \{0_V\} \)

By definition \( f : V \to W \) is injective if \( f(v_1) = f(v_2) \Rightarrow v_1 = v_2 \). The kernel of \( f \) on the other hand is defined by \( \ker(f) = \{v \in V \mid f(v) = 0_W\} \).

\[ \Rightarrow \] Suppose \( f \) is injective. Let \( v \in \ker(f) \). We always have \( f(0_V) = 0_W \), for \( 0_W = \beta \cdot f(v) = f(0_V) \). So now, by injectivity

\[ f(v) = 0_W \quad \text{and} \quad f(0_V) = 0_W \Rightarrow v = 0_V \]

Thus, in fact, the only vector in \( \ker(f) \) is \( 0_V \).

\[ \Leftarrow \] Suppose \( \ker(f) = \{0_V\} \). If \( f(v_1) = f(v_2) \), then

\[ 0_W = f(v_1) - f(v_2) \Rightarrow f(v_1 - v_2) = 0_W \Rightarrow v_1 - v_2 \in \ker(f) = \{0_V\} \]

\[ \Rightarrow v_1 = v_2 \]

and so \( f \) is injective.

(b) Prove that \( f \) is surjective if and only if \( \dim(\operatorname{Im}(f)) = \dim(W) \).

\[ \Rightarrow \] This way is easy. If \( f \) is surjective then, by definition, \( \operatorname{Im}(f) = W \), and so \( \dim(\operatorname{Im}(f)) = \dim(W) \).
3. Consider the mapping \( T \).

(a) Show that \( T \) is a linear transformation.

(b) Find the matrix corresponding to \( T \) and the natural bases of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

(c) What is the kernel of this linear transformation.

(c) Suppose \( f \) is bijective. Then it is injective and surjective. Hence, \( \ker (f) = \{0_V\} \) by part (a) and by part (b), \( \text{Im} (f) = W \).

\[ \text{dim} (V) = \text{dim} (\text{Im} (f)) + \text{dim} (\ker (f)) \]

and so

\[ \text{dim} (V) = \text{dim} (W) + 0 \Rightarrow \text{dim} V = \text{dim} W \]

and so \( \text{Im} (f) \subset W \) has the same dimension as \( W \), and so \( \text{Im} (f) = W \). Hence \( f \) is surjective. Since \( f \) is both injective and surjective it is a bijection.

3. Consider the mapping \( T : \mathbb{R}^2 \to \mathbb{R}^3 \)

\( T ([x_1, x_2]) = [x_1 - x_2, x_1 + x_2, x_1 - 2x_2] \)

(a) Show that \( T \) is a linear transformation.

- Let us write two arbitrary elements of \( \mathbb{R}^2 \) as \([x_1, x_2], [y_1, y_2]\). Then

\[ \alpha [x_1, x_2] + \beta [y_1, y_2] = [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2] \]

and so

\[ T (\alpha [x_1, x_2] + \beta [y_1, y_2]) = T ([\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2]) \]

\[ = [[\alpha x_1 + \beta y_1] - (\alpha y_2 + \beta y_2), (\alpha x_1 + \beta y_1) - (\alpha y_2 + \beta y_2), (\alpha x_1 + \beta y_1) - 2(\alpha y_2 + \beta y_2)]] \]

\[ = \alpha [x_1 - x_2, x_1 + x_2, x_1 - 2x_2] + \beta [y_1 - y_2, y_1 + y_2, y_1 - 2y_2] \]

\[ = \alpha T ([x_1, x_2]) + \beta T ([y_1, y_2]) \]

Since \( T \) preserves arbitrary linear combinations of elements, it is a linear transformation. \( \square \)

(b) Find the matrix corresponding to \( T \) and the natural bases of \( B = \{[1, 0], [0, 1]\} \) and \( B' = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \) of, respectively, \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

- The matrix \( T_{B,B'} \) corresponding to the linear transformation \( T \) is formed by using the components of the \( T (e_i) \) as the \( i^{th} \) column. We have

\[ T ([1, 0]) = [1 - 0, 1 + 0, 1 - 2 \cdot 0] = [1, 1, 1] \]

\[ T ([0, 1]) = [0 - 1, 0 + 1, 0 - 2 \cdot 1] = [-1, 1, -2] \]

Thus,

\[ T_{B,B'} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} \]

Notice that

\[ T_{B,B'} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix} \]

replicates the formula for \( T \) so long as we interpret the vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) as column vectors.

(c) What is the kernel of this linear transformation.
• The kernel of the transformation will correspond to the null space of the matrix \( T_{B,B'} \); i.e., the solution set of \( T_{B,B'} \mathbf{x} = \mathbf{0} \). A basis for this solution set can be found using our augmented matrix method of solving such a homogeneous linear system.

\[
\begin{bmatrix}
1 & -1 & 0 \\
1 & 1 & 0 \\
1 & -2 & 0 \\
\end{bmatrix}
\rightarrow
\text{R.R.E.F.}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\Rightarrow
\begin{cases}
x_1 = 0 \\
x_2 = 0 \\
\end{cases}
\]

So,

\[ \text{Ker}(T) = \text{NullSp}(T_{B,B'}) = \text{span}\{[0,0]\} = \{0_R \} \]

(d) What is the range of this linear transformation.

• The range of \( T \) coincides with the span of the columns of \( T_{B,B'} \). To get this span, we can covert rows into columns and thereby obtain the transpose matrix \( T^t_{B,B'} \). A basis for the row space of \( T^t_{B,B'} \) will be a basis for the column space of \( T_{B,B'} \). We can obtain a basis for \( \text{RowSp}(T^t_{B,B'}) \) using row reduction:

\[
T^t_{B,B'} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}
\]

So,

\[ \text{RowSp}(T^t_{B,B'}) = \text{span}\{[1,1,1],[0,2,-1]\} \Rightarrow \text{ColSp}(T_{B,B'}) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right) \]

Thus, vectors in the range of \( T \) will be vectors in \( \mathbb{R}^3 \) of the form

\[ s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} s+s+2t \end{bmatrix} \]

4. Let \( \mathcal{P}_3 \) be the vector space of polynomials of degree \( \leq 3 \) with natural basis \{\( x^3, x^2, x, 1 \)\}. Find the matrix \( T_{B,B} \) corresponding to the linear transformation

\[ T : \mathcal{P}_3 \rightarrow \mathcal{P}_3, \quad p \rightarrow 2x \frac{d}{dx}p + p \]

and the basis \( B \) (same basis for the domain and codomain of \( T \)).

• The matrix \( T_{B,B} \) is obtained by using the coefficients of \( T(p_i) \) with respect to \( B = \{ p_1, p_2, p_3, p_4 \} = \{ x^3, x^2, x, 1 \} \) as the columns of a matrix.

\[
T(p_1) = T(x^3) = \left(2x \frac{d}{dx}+1\right) x^3 = 2x(3x^2) + x^3 = 7 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 0 \cdot 1 \quad \leftrightarrow \quad \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
T(p_2) = T(x^2) = \left(2x \frac{d}{dx}+1\right) x^2 = 2x(2x) + x^2 = 0 \cdot x^3 + 5 \cdot x^2 + 0 \cdot x + 0 \cdot 1 \quad \leftrightarrow \quad \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}
\]

\[
T(p_3) = T(x) = \left(2x \frac{d}{dx}+1\right) x = 2x(1) + x = 0 \cdot x^3 + 0 \cdot x^2 + 3 \cdot x + 0 \cdot 1 \quad \leftrightarrow \quad \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}
\]

\[
T(p_4) = T(1) = \left(2x \frac{d}{dx}+1\right) 1 = 0 + 1 = 0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 1 \cdot 1 \quad \leftrightarrow \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]
Thus, 

\[
T_{B,B} = \begin{bmatrix}
7 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

5. Suppose \( f : V \to W \) is a linear transformation and \( S \) is a subspace of \( W \) contained in \( \text{Im}(f) \). Prove that \( f^{-1}(S) \equiv \{ v \in V \mid f(v) \in S \} \) is a subspace of \( V \).

- Suppose \( v_1 \) and \( v_2 \in f^{-1}(S) \). We aim to show that any linear combination \( \alpha v_1 + \beta v_2 \) of \( v_1 \) and \( v_2 \) will also be in \( f^{-1}(S) \). Now now

  \[
  v_1 \in f^{-1}(S) \Rightarrow \exists s_1 \in S \text{ such that } f(v_1) = s_1
  \]
  \[
  v_2 \in f^{-1}(S) \Rightarrow \exists s_2 \in S \text{ such that } f(v_2) = s_2
  \]

  Now consider the linear combination \( \alpha v_1 + \beta v_2 \)

  \[
  f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2) \quad \text{because } f \text{ is a linear transformation}
  \]

  \[
  = \alpha s_1 + \beta s_2 \in S \quad \text{because } s_1, s_2 \in S \text{ and } S \text{ is a subspace}
  \]

  Hence, \( \alpha v_1 + \beta v_2 \in f^{-1}(S) \).

8. Let \( S \) be the subspace of \( \mathbb{R}^3 \) spanned by \([1,0,0]\) and \([0,1,0]\). Identify let \( v_1 = [1,-1,3] \) and let \( v_2 = [2,3,1] \). Determine \([v_1]_S + [v_2]_S\) explicitly (it has to be some hyperplane in the direction of \( S \) inside \( \mathbb{R}^3 \)).

- We have

  \[
  [v_1]_S + [v_2]_S = [v_1 + v_2]_S = [[1,2,4]]_S
  \]

  \[
  = \{ x \in \mathbb{R}^3 \mid x = [1,2,4] + c_1 [1,0,0] + c_2 [0,1,0] \quad , \quad c_1, c_2 \in \mathbb{R} \}
  \]

  \[
  = \{ [1 + c_1, 2 + c_1, 4] \mid c_1, c_2 \in \mathbb{R} \}
  \]

  \[
  = \text{hyperplane perpendicular to the } z \text{-axis and intersecting the } z \text{-axis at } [0,0,4]
  \]