MATH 4063-5023
Solutions to Homework Set 4

1. Let $\mathcal{P}$ be the vector space of polynomials with indeterminant x.Which of the following mappings are linear transformations from $\mathcal{P}$ to itself

(a) $T : p \to xp$

- Let p_1, p_2 be two polynomials and let $\alpha, \beta \in \mathbb{F}$. We have

$$\begin{aligned} p_1 &= a_n x^n + \cdots + a_1 x + a_0 \\ p_2 &= b_m x^m + \cdots + b_1 x + b_0 \end{aligned}$$

If $n \neq m$ we can anyway replace the polynomial of lower degree (e.g. say it's p_2) with its equivalent expression

$$p_2 = 0 \cdot x^n + 0 \cdot x^{n-1} + \cdots + 0 \cdot x^{m+1} + b_m x^m + \cdots + b_1 x + b_0$$

So we can without loss of generality write

$$\begin{aligned} p_1 &= a_n x^n + \cdots + a_1 x + a_0 \\ p_2 &= b_n x^n + \cdots + b_1 x + a_0 \end{aligned}$$

for a pair of arbitary polynomials. Then by the definition of scalar multiplication and addition in $\mathcal{P}$ we'll have

$$\alpha p_1 + \beta p_2 = (\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)$$

and so

$$\begin{aligned} T(\alpha p_1 + \beta p_2) &= x\left[(\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)\right] \\ &= (\alpha a_n + \beta b_n) x^{n+1} + \cdots + (\alpha a_1 + \beta b_1) x^2 + (\alpha a_0 + \beta b_0) x \\ &= (\alpha a_n) x^{n+1} + \cdots + (\alpha a_1) x^2 + (\alpha a_0) x \\ &\quad + (\beta b_n) x^{n+1} + \cdots + (\beta b_1) x^2 + (\beta b_0) x \\ &= \alpha x (a_n x^n + a_1 x + a_0) + \beta x (b_n x^n + \cdots + b_1 x + b_0) \\ &= \alpha T(p_1) + \beta T(p_2) \end{aligned}$$

Since T preseves arbitary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. $\square$

(b) $T : p \to 2p$

- Using the same setup as in preceding problem, we compute

$$\begin{aligned} T(\alpha p_1 + \beta p_2) &= 2\left[(\alpha a_n + \beta b_n) x^n + \cdots + (\alpha a_1 + \beta b_1) x + (\alpha a_0 + \beta b_0)\right] \\ &= (2\alpha a_n + 2\beta b_n) x^n + \cdots + (2\alpha a_1 + 2\beta b_1) x + (2\alpha a_0 + 2\beta b_0) \\ &= \alpha 2 (a_n x^n + a_1 x + a_0) + \beta 2 (b_n x^n + \cdots + b_1 x + b_0) \\ &= \alpha T(p_1) + \beta T(p_2) \end{aligned}$$

Since T preseves arbitary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. $\square$

(c) $T : p \to \frac{dp}{dx} + 2p$

- Here we won't be so explicity as in parts (a) and (b); we'll simply use the facts that differentiation operates term by term and commutes with scalar multiplication.

$$\begin{aligned} T(\alpha p_1 + \beta p_2) &= \left(\frac{d}{dx} + 2\right)(\alpha p_1 + \beta p_2) \\ &= \alpha \frac{dp_1}{dx} + 2\alpha p_1 + \beta \frac{dp_2}{dx} + 2\beta p_2 \\ &= \alpha \left(\frac{d}{dx} + 2\right) p_1 + \beta \left(\frac{d}{dx} + 2\right) p_2 \\ &= \alpha T(p_1) + \beta T(p_2) \end{aligned}$$

Since T preseves arbitary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. □

(d) $T : p \to \int_0^1 p(x)\, dx$

- Here we'll just use the facts that we can integrate term by term and pull constants through integral signs.

$$\begin{aligned} T(\alpha p_1 + \beta p_2) &= \int_0^1 (\alpha p_1 + \beta p_2)(x)\, dx \\ &= \int_0^1 (\alpha p_1(x) + \beta p_2(x))\, dx \\ &= \int_0^1 \alpha p_1(x)\, dx + \int_0^1 \beta p_2(x)\, dx \\ &= \alpha \int_0^1 p_1(x)\, dx + \beta \int_0^1 p_2(x)\, dx \\ &= \alpha T(p_1) + \beta T(p_2) \end{aligned}$$

Since T preseves arbitary linear combinations of elements of $\mathcal{P}$, it is a linear transformation. □

2. Suppose $f : V \to W$ is a linear transformation.

(a) Prove that f is injective if and only if $\ker(f) = \{\mathbf{0}_V\}$

- By definition $f : V \to W$ is injective if $f(v_1) = f(v_2) \quad \Rightarrow \quad v_1 = v_2$. The kernel of f on the other hand is defined by $\ker(f) = \{v \in V \mid f(v) = \mathbf{0}_W\}$.

  $\Rightarrow$ Suppose f is injective. Let $v \in \ker(f)$. We always have $f(\mathbf{0}_V) = \mathbf{0}_W$, for $\mathbf{0}_W = 0_{\mathbb{F}} \cdot f(v) = f(0_{\mathbb{F}} \cdot v) = f(\mathbf{0}_V)$. So now, by injectivity

$$f(v) = \mathbf{0}_W \quad \text{and} \quad f(\mathbf{0}_V) = \mathbf{0}_W \quad \Rightarrow \quad v = \mathbf{0}_V$$

  Thus, in fact, the only vector in $\ker(f)$ is $\mathbf{0}_V$.

  $\Longleftarrow$ Suppose $\ker(f) = \{\mathbf{0}_V\}$. If $f(v_1) = f(v_2)$, then

$$\begin{aligned} \mathbf{0}_W &= f(v_1) - f(v_2) \quad \Rightarrow \quad f(v_1 - v_2) = \mathbf{0}_W \quad \Rightarrow \quad v_1 - v_2 \in \ker\{f\} = \{\mathbf{0}_V\} \\ &\Rightarrow \quad v_1 - v_2 = \mathbf{0}_V \\ &\Rightarrow \quad v_1 = v_2 \end{aligned}$$

  and so f is injective.

(b) Prove that f is surjective if and only if $\dim(\mathrm{Im}(f)) = \dim(W)$.

- $\Rightarrow$ This way is easy. If f is surjective then, by definition, $\mathrm{Im}(f) = W$, and so $\dim(\mathrm{Im}(f)) = \dim(W)$.

$\Longleftarrow$ $\quad \mathrm{Im}(f) = \{w \in W \mid w = f(v) \text{ for some } v \in V\}$ is defined as a subspace of W. We know from way back that if a subspace of a vector space has the same dimension as the vector space in which it lives, then it must in fact coincide with the parent vector space. So

$$\dim(\mathrm{Im}(f)) = \dim(W) \text{ and } \mathrm{Im}(f) \subset W \quad \Rightarrow \quad \mathrm{Im}(f) = W \quad \Rightarrow \quad f \text{ is surjective.}$$

(c) Prove that f is bijective if and only it $\dim(V) = \dim(W)$ and $\ker(f) = \{\mathbf{0}_V\}$.

- $\Rightarrow$ Suppose f is bijective. Then it is injective and surjective. Hence, $\ker(f) = \{\mathbf{0}_V\}$ by part (a) and by part (b), $\mathrm{Im}(f) = W$.From Theorem 12.1

$$\dim(V) = \dim(\mathrm{Im}(f)) + \dim(\ker(f))$$

and so

$$\dim(V) = \dim(W) + 0 \quad \Rightarrow \quad \dim V = \dim W$$

$\Longleftarrow$ $\quad \dim(V) = \dim(W)$ and $\ker(f) = \{\mathbf{0}_V\}$. By part (a), f is injective, and so by Theorem 12.1

$$\dim(V) = \dim(\mathrm{Im}(f)) + 0$$

and so $\mathrm{Im}(f) \subset W$ has the same dimension as W, and so $\mathrm{Im}(f) = W$, Hence f is surjective. Since f is both injective and surjective it is a bijection.

3. Prove that the composition $f \circ g$ of two linear transformations is a linear transformation.

- Let $f : V \to W$ and $g : U \to V$ be linear transformations. Then $f \circ g$ is a mapping from U to W. Let $u_1, u_2 \in U$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned}
(f \circ g)(\alpha u_1 + \beta u_2) &\equiv f(g(\alpha u_1 + \beta u_2)) \\
&= f(\alpha g(u_1) + \beta g(u_2)) \quad \text{since } g \text{ is a linear transformation} \\
&= \alpha f(g(u_1)) + \beta f(g(u_2)) \quad \text{since } f \text{ is a linear transformation} \\
&= \alpha (f \circ g)(u_1) + \beta (f \circ g)(u_2) \\
&\Rightarrow \quad f \circ g \text{ is a linear transformation}
\end{aligned}$$

4. Consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3 \quad T([x_1, x_2]) = [x_1 - x_2, x_1 + x_2, x_1 - 2x_2]$

(a) Show that T is a linear transformation.

- Let us write two arbitrary elements of $\mathbb{R}^2$ as $[x_1, x_2]$, $[y_1, y_2]$. Then

$$\alpha[x_1, x_2] + \beta[y_1, y_2] = [\alpha x_1 + \beta y_1, \alpha y_2 + \beta y_2]$$

and so

$$\begin{aligned}
T(\alpha[x_1, x_2] + \beta[y_1, y_2]) &= T([\alpha x_1 + \beta y_1, \alpha y_2 + \beta y_2]) \\
&= [(\alpha x_1 + \beta y_1) - (\alpha y_2 + \beta y_2), (\alpha x_1 + \beta y_1) - (\alpha y_2 + \beta y_2), (\alpha x_1 + \beta y_1) - 2(\alpha y_2 + \beta y_2)] \\
&= \alpha[x_1 - x_2, x_1 + x_2, x_1 - 2x_2] + \beta[y_1 - y_2, y_1 + y_2, y_1 - 2y_2] \\
&= \alpha T([x_1, x_2]) + \beta T([y_1, y_2])
\end{aligned}$$

Since T preseves arbitary linear combinations of elements, it is a linear transformation. $\square$

(b) Find the matrix corresponding to T and the natural bases of $B = \{[1,0], [0,1]\}$ and $B' = \{[1,0,0], [0,1,0], [0,0,1]\}$ of, respectively, $\mathbb{R}^2$ and $\mathbb{R}^3$.

- The matrix $\mathbf{T}_{BB'}$ corresponding to the linear transformation T is formed by using the components of the $T(\mathbf{e}_i)$ as the i^{th} column. We have

$$\begin{aligned} T([1,0]) &= [1-0,1+0,1-2\cdot 0] = [1,1,1] \\ T([0,1]) &= [0-1,0+1,0-2\cdot 1] = [-1,1,-2] \end{aligned}$$

Thus,

$$\mathbf{T}_{B,B'} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Notice that

$$\mathbf{T}_{B,B'} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

replicates the formula for T so long as we interpret the vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as column vectors.

(c) What is the kernel of this linear transformation.

- The kernel of the transformation will correpond to the null space of the matrix $\mathbf{T}_{B,B'}$; i.e, the solution set of $\mathbf{T}_{B,B'}\mathbf{x} = \mathbf{0}$. A basis for this solution set can be found using our augmented matrix method of solving such a homogeneous linear system.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{R.R.E.F.}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \left\{ \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array} \right.$$

So

$$Ker(T) = NullSp(\mathbf{T}_{B,B'}) = span([0,0]) = \{\mathbf{0}_{\mathbb{R}^2}\}$$

(d) What is the range of this linear transformataion.

- The range of T coincides with the span of the columns of $\mathbf{T}_{B,b'}$. To get this span, we can covert rows into columns and thereby obtain the transpose matrix $\mathbf{T}^t_{B,B'}$. A basis for the row space of $\mathbf{T}^t_{B,B'}$ will be a basis for the column space of $\mathbf{T}_{B,B'}$. We can obtain a basis for $RowSp\left(\mathbf{T}^t_{b,b'}\right)$ using row reduction:

$$\mathbf{T}^t_{B,B'} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

So

$$RowSp\left(\mathbf{T}^t_{B,B'}\right) = span([1,1,1],[0,2,-1]) \quad \Rightarrow \quad ColSp(T_{B,B'}) = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right)$$

Thus, vectors in the range of T will be vectors in $\mathbb{R}^3$ of the form

$$s[1,1,1] + t[0,2,-1] = [s, s+2t, s-t]$$

5. Let $\mathcal{P}_3$ be the vector space of polyonomials of degree ≤ 3 with natural basis $\{x^3, x^2, x, 1\}$. Find the matrix $T_{B,B}$ corresponding to the linear transformation

$$T : \mathcal{P}_3 \rightarrow \mathcal{P}_3 \quad , \quad p \rightarrow 2x\frac{d}{dx}p + p$$

and the basis B (same basis for the domain and codomain of T).

- The matrix $\mathbf{T}_{B,B}$ is obtained by using the coefficients of $T(p_i)$ with respect to $B = \{p_1, p_2, p_3, p_4\} = \{x^3, x^2, x, 1\}$ as the columns of a matrix.

$$\begin{aligned}
T(p_1) &= T(x^3) = \left(2x\frac{d}{dx} + 1\right)x^3 = 2x(3x^2) + x^3 = 7\cdot x^3 + 0\cdot x^2 + 0\cdot x + 0\cdot 1 \longleftrightarrow \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
T(p_2) &= T(x^2) = \left(2x\frac{d}{dx} + 1\right)x^2 = 2x(2x) + x^2 = 0\cdot x^3 + 5\cdot x^2 + 0\cdot x + 0\cdot 1 \longleftrightarrow \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} \\
T(p_3) &= T(x) = \left(2x\frac{d}{dx} + 1\right)x = 2x(1) + x = 0\cdot x^3 + 0\cdot x^2 + 3\cdot x + 0\cdot 1 \longleftrightarrow \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \\
T(p_4) &= T(1) = \left(2x\frac{d}{dx} + 1\right)1 = 0 + 1 = 0\cdot x^3 + 0\cdot x^2 + 0\cdot x + 1\cdot 1 \longleftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

Thus,

$$\mathbf{T}_{B,B} = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□

6. Suppose $f : V \to W$ is a linear transformation and S is a subspace of W contained in $\mathrm{Im}(f)$. Prove that

$$f^{-1}(S) \equiv \{v \in V \mid f(v) \in S\}$$

is a subspace of V.

- Let $v_1, v_2 \in f^{-1}(S)$. Then, by definition, there exists vectors $s_1, s_2 \in S$ such that $f(v_1) = s_1$ and $f(v_2) = s_2$. Now let $\alpha, \beta \in \mathbb{F}$ and consider $f(\alpha v_1 + \beta v_2)$. We have

$$\begin{aligned}
f(\alpha v_1 + \beta v_2) &= \alpha f(v_1) + \beta f(v_2) \\
&= \alpha s_1 + \beta s_2 \\
&\in S \qquad \text{since } S \text{ is a subspace of } W
\end{aligned}$$

Thus, $\alpha v_1 + \beta v_2$ is in $f^{-1}(S)$, and so $f^{-1}(S)$ is a subset of V that is closed under linear combinations; hence $f^{-1}(S)$ is a subspace of V. □

7. Let S be a subspace of a vector space V over a field $\mathbb{F}$ and let V/S be the corresponding quotient space:

$$V/S := \{v + S \mid v \in V\}$$

where

$$v + S := \{v' \in V \mid v' = v + s \quad \text{for some } s \in S\}$$

Let addition and scalar multiplication of elements of V/S be defined by

$$\begin{aligned}
+ &: V/S \times V/S \to V/S \quad ; \quad (v + S) + (w + S) := (v + w + S) \\
* &: \mathbb{F} \times V/S \to V/S \quad ; \quad \lambda(v + S) := (\lambda v + S)
\end{aligned}$$

Show that V/S is a vectors space over $\mathbb{F}$ (i.e., verify all 8 axioms for a vector space).

- – Commutativity of addition:

$$\begin{aligned}(v+S)+(w+S) &\equiv (v+w)+S && \text{by definition of addition in } V/S\\ &= (w+v)+S && \text{because addition is commutative in } V\\ &\equiv (w+S)+(v+S) && \text{by definition of addition in } V/S\end{aligned}$$

  – Associativity of addition:

$$\begin{aligned}((v+S)+(w+S))+(u+S) &\equiv ((v+w)+S)+(u+S)\\ &\equiv ((v+w)+u)+S\\ &= (v+(w+u))+S \quad \text{by associativity of addition in } V\\ &\equiv (v+S)+((w+u)+S)\\ &\equiv (v+S)+((w+S)+(u+S))\end{aligned}$$

  – Existence of additive identity.
  The additive identity in V/S is $0_V + S = S$. We have

$$(v+S)+(0_V+S) = (v+0_V)+S = v+S$$

  – Existence of additive inverses.
  Let $v+S \in V/S$. Then $-v+S$ is also in V/S and

$$(v+S)+(-v+S) = (v+(-v))+S = 0_V+S = 0_{V/S}$$

  – Compatibility of scalar multiplication.

$$\begin{aligned}(\lambda\mu)\cdot(v+S) &\equiv ((\lambda\mu)\,v+S)\\ &= (\lambda\,(\mu v)+S) \quad \text{compatibility of scalar multiplication in } V\\ &\equiv \lambda\cdot(\mu v+S)\\ &= \lambda\cdot(\mu\cdot(v+S))\end{aligned}$$

  – Distributivity of addition of scalars

$$\begin{aligned}(\lambda+\mu)\cdot(v+S) &\equiv (\lambda+\mu)\,v+S\\ &= (\lambda v+\mu v)+S \quad \text{distributivity of addition of scalars in } V\\ &\equiv (\lambda v+S)+(\mu v+S)\\ &\equiv \lambda\cdot(v+S)+\mu\cdot(v+S)\end{aligned}$$

  – Distributivity of scalar multiplication over vector addition

$$\begin{aligned}\lambda\cdot((v+S)+(u+S)) &\equiv \lambda\cdot((v+u)+S)\\ &\equiv (\lambda\,(v+u)+S)\\ &= (\lambda v+\lambda u)+S\\ &= (\lambda v+S)+(\lambda u+S)\\ &\equiv \lambda\cdot(v+S)+\lambda\cdot(u+S)\end{aligned}$$

  – $1_{\mathbb{F}}$ acts as identiy operator w.r.t. scalar multiplication.

$$\begin{aligned}1_{\mathbb{F}}\cdot(v+S) &\equiv (1_{\mathbb{F}}\cdot v)+S\\ &= v+S\end{aligned}$$