1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}^n$ denote the set of $n$-tuples of elements of $\mathbb{F}$, with operations of scalar multiplication and vector addition defined by

$$\lambda \cdot [\alpha_1, \ldots, \alpha_n] := [\lambda a_1, \ldots, \lambda a_n], \quad \text{for all } \lambda \in \mathbb{F} \text{ and all } [\alpha_1, \ldots, \alpha_n] \in \mathbb{F}^n$$

$$[\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] := [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n], \quad \text{for all } [\alpha_1, \ldots, \alpha_n] \text{ and } [\beta_1, \ldots, \beta_n] \in \mathbb{F}^n$$

Check that $\mathbb{F}^n$ satisfies all the axioms of a vector space over $\mathbb{F}$.

- There are 8 axioms to check. We’ll check them one by one, constantly using the hypothesis that $\mathbb{F}$ is a field (and so obeys the 9 axioms of a field (see Definition 1.7 of Lecture 1).
  
  (i) Commutativity of Vector Addition

$$[\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] := [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n]$$

$$= [\beta_1 + \alpha_1, \ldots, \beta_n + \alpha_n] \quad \text{because addition in } \mathbb{F} \text{ is commutative}$$

$$= [\beta_1, \ldots, \beta_n] + [\alpha_1, \ldots, \alpha_n]$$

(ii) Associativity of Vector Addition

$$([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) + [\gamma_1, \ldots, \gamma_n] := [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] + [\gamma_1, \ldots, \gamma_n]$$

$$= [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] + [\gamma_1, \ldots, \gamma_n]$$

$$= [(\alpha_1 + \beta_1) + \gamma_1, \ldots, (\alpha_n + \beta_n) + \gamma_n]$$

$$= [\alpha_1 + \gamma_1, \ldots, \alpha_n + \gamma_n] + [\beta_1, \ldots, \beta_n + \gamma_n]$$

$$= [\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] + [\gamma_1, \ldots, \gamma_n]$$

(iii) Existence of Additivity Identity.

Set $0_{\mathbb{F}^n} = [0_\mathbb{F}, \ldots, 0_\mathbb{F}]$. Then for any vector $[\alpha_1, \ldots, \alpha_n] \in \mathbb{F}^n$

$$[\alpha_1, \ldots, \alpha_n] + 0_{\mathbb{F}^n} = [\alpha_1, \ldots, \alpha_n] + [0_\mathbb{F}, \ldots, 0_\mathbb{F}]$$

$$= [\alpha_1 + 0_\mathbb{F}, \ldots, \alpha_n + 0_\mathbb{F}]$$

$$= [\alpha_1, \ldots, \alpha_n] \quad \text{because } 0_\mathbb{F} \text{ is the additive identity in } \mathbb{F}$$

(iv) Existence of Additive Inverses

We need to show that for each vector $\mathbf{v} \in \mathbb{F}^n$ there exists another vector $(-\mathbf{v}) \in \mathbb{F}^n$ such that $\mathbf{v} + (-\mathbf{v}) = 0_{\mathbb{F}^n}$. Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$ and set $-\mathbf{v} = [-\alpha_1, \ldots, -\alpha_n]$. The latter expression makes sense since each element $\alpha_i \in \mathbb{F}$ has an additive inverse. Then

$$\mathbf{v} + (-\mathbf{v}) = [\alpha_1, \ldots, \alpha_n] + [-\alpha_1, \ldots, -\alpha_n]$$

$$= [\alpha_1 + (-\alpha_1), \ldots, \alpha_n + (-\alpha_n)]$$

$$= [0_\mathbb{F}, \ldots, 0_\mathbb{F}]$$

$$= 0_{\mathbb{F}^n}$$

(v) Associativity and Compatibility of Scalar Multiplication

We need to show that if $\lambda, \mu \in \mathbb{F}$, then $\lambda (\mu \mathbf{v}) = (\lambda \mu) \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$. Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$. Then

$$\lambda (\mu \mathbf{v}) = \lambda (\mu [\alpha_1, \ldots, \alpha_n])$$

$$= \lambda (\mu a_1, \ldots, \mu a_n)$$

$$= [\lambda (\mu a_1), \ldots, \lambda (\mu a_n)]$$

$$= [(\lambda \mu) a_1, \ldots, (\lambda \mu) a_n] \quad \text{by associativity of multiplication in } \mathbb{F}$$

$$= (\lambda \mu) [\alpha_1, \ldots, \alpha_n]$$

$$= (\lambda \mu) \mathbf{v}$$
(vi) Distributivity of Scalar Multiplication over Addition of Scalars
We need to show that if \( \lambda, \mu \in \mathbb{F} \) and \( \mathbf{v} \in \mathbb{F}^n \) that \( (\lambda + \mu) \mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v} \). Let \( \mathbf{v} = [\alpha_1, \ldots, \alpha_n] \).
Then
\[
(\lambda + \mu) \mathbf{v} = (\lambda + \mu) [\alpha_1, \ldots, \alpha_n] \\
= [\lambda \alpha_1 + \mu \alpha_1, \ldots, \lambda \alpha_n + \mu \alpha_n] \\
= \lambda [\alpha_1, \ldots, \alpha_n] + \mu [\alpha_1, \ldots, \alpha_n] \\
= \lambda \mathbf{v} + \mu \mathbf{v}
\]

(vii) Distributivity of Scalar Multiplication over Vector Addition
We need to show that if \( \lambda \in \mathbb{F} \) and \([\alpha_1, \ldots, \alpha_n], [\beta_1, \ldots, \beta_n] \in \mathbb{F}^n\) then \( \lambda ([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) = \lambda [\alpha_1, \ldots, \alpha_n] + \lambda [\beta_1, \ldots, \beta_n]\)
\[
\lambda ([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) \\
= \lambda [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] \\
= [\lambda \alpha_1 + \lambda \beta_1, \ldots, \lambda \alpha_n + \lambda \beta_n] \\
= \lambda [\alpha_1, \ldots, \alpha_n] + \lambda [\beta_1, \ldots, \beta_n]
\]

(viii) Scalar Multiplication by \( 1\mathbb{F} \)
We have for any \( \mathbf{v} = [\alpha_1, \ldots, \alpha_n] \in \mathbb{F}^n \)
\[
1\mathbb{F} [\alpha_1, \ldots, \alpha_n] := [1\mathbb{F} \alpha_1, \ldots, 1\mathbb{F} \alpha_n] = [\alpha_1, \ldots, \alpha_n]
\]
and so scalar multiplication by the multiplicative identity \( 1\mathbb{F} \) in \( \mathbb{F} \) acts trivially. \( \Box \)

2. Let \( C^1(\mathbb{R}) \) be the set of continuous, differentiable functions on the real line with values in \( \mathbb{R} \). Define scalar multiplication and vector addition on \( C(\mathbb{R}) \) by
\[
(\lambda \cdot f)(x) := \lambda f(x) \quad , \quad \forall \lambda \in \mathbb{R} \quad , \quad \forall f \in C^1(\mathbb{R}) ;
\]
\[
(f + g)(x) := f(x) + g(x) \quad , \quad \forall f, g \in C^1(\mathbb{R}) .
\]
Check that \( C^1(\mathbb{R}) \) satisfies the axioms for a vector space over \( \mathbb{R} \).

- Below we check the axioms. In the computations below we constant use the circumstance that two functions in \( C^1(\mathbb{R}) \) coincide if they have exactly the same value at each point \( x \in \mathbb{R} \).

  (i) Commutativity of Vector Addition
\[
(f + g)(x) \quad := \quad f(x) + g(x) \\
= g(x) + f(x) \quad \text{because addition in } \mathbb{R} \text{ is commutative} \\
= (g + f)(x)
\]

  (ii) Associativity of Vector Addition
\[
((f + g) + h)(x) \quad := \quad (f + g)(x) + h(x) \\
= f(x) + g(x) + h(x) \quad \text{because addition in } \mathbb{R} \text{ is associative} \\
= (f + (g + h))(x)
\]

  (iii) Existence of Additivity Identity.
Set $0_{C^1(\mathbb{R})}$ to be the function on $\mathbb{R}$ with constant value 0. Then for any function $f \in C^1(\mathbb{R})$
\[
(f + 0_{C^1(\mathbb{R})}) (x) \quad = \quad f (x) + 0_{C^1(\mathbb{R})} (x) \\
= \quad f (x) + 0 \\
= \quad f (x)
\]

(iii). We have for any $f \in C^1(\mathbb{R})$ acts trivially.

We need to show that the polynomial functions are closed under the operations of taking linear combinations. Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + \cdots + b_1 x + b_0$. Without loss of generality, we can assume $m = n$ (For example, if $m < n$ we can add $0 \cdot x^n + 0 \cdot x^{n-1} + \cdots + 0 \cdot x^{m+1}$ to $g$ without changing its values as a function.). Then
\[
(\alpha f + \beta g) (x) \quad = \quad (a_n x^n + \cdots + a_1 x + a_0) + (b_m x^m + \cdots + b_1 x + b_0) \\
= \quad (\alpha_n + \beta_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0)
\]

Since the expression on the far right is a polynomial function, we conclude that the subset of polynomial functions is closed under scalar multiplication and vector addition; hence, it is a subspace of $C^1(\mathbb{R})$.

3. Determine which of the following subsets are subspaces of $C^1(\mathbb{R})$

(a) The set of polynomial functions in $C^1(\mathbb{R})$.
- The set of polynomial functions on $\mathbb{R}$ form a subset of $C^1(\mathbb{R})$, to show that it is in fact a subspace just need to show that the polynomial functions are closed under the operations of taking linear combinations. Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + \cdots + b_1 x + b_0$. Without loss of generality, we can assume $m = n$ (For example, if $m < n$ we can add $0 \cdot x^n + 0 \cdot x^{n-1} + \cdots + 0 \cdot x^{m+1}$ to $g$ without changing its values as a function.). Then
\[
(\alpha f + \beta g) (x) \quad = \quad (a_n x^n + \cdots + a_1 x + a_0) + (b_m x^m + \cdots + b_1 x + b_0) \\
= \quad (\alpha_n + \beta_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0)
\]

Since the expression on the far right is a polynomial function, we conclude that the subset of polynomial functions is closed under scalar multiplication and vector addition; hence, it is a subspace of $C^1(\mathbb{R})$. 

□
4. Is the intersection of two subspaces a subspace (prove your answer)?
   - Yes. Let $W, U$ be two subspaces of a vector space $V$. The intersection of $W$ and $U$ is
     $$ W \cap U = \{ v \in V \mid v \in W \text{ and } v \in U \}. $$

     Let $u, v$ be any two vectors in $W \cap U$, and let $\alpha, \beta \in \mathbb{R}$. Consider the linear combination $\alpha u + \beta v$. Because, $u, v \in W \cap U$, in particular, both $u$ and $v$ live in the subspace $W$. Since $W$ is a subspace, we have $\alpha u + \beta v \in W$. On the other hand, both $u$ and $v$ live in $U$, and so because $U$ is a subspace, $\alpha u + \beta v$ lies in $U$. Thus, $\alpha u + \beta v$ lies in both $W$ and $U$ and so it lies in $W \cap U$. Thus, the intersection of two subspaces is closed under scalar multiplication and vector addition; hence it too is a subspace.

5. Is the union of two subspaces a subspace (explain your answer)?
   - No. A single counter-example can justify this claim. Consider the $x$ and $y$ axes of the usual Cartesian plane $\mathbb{R}^2$.
     \[
     \ell_x = \{ [x, 0] \mid x \in \mathbb{R} \} \\
     \ell_y = \{ [0, y] \mid y \in \mathbb{R} \}
     \]

     Each axis is a 1-dimensional subspace of $\mathbb{R}^2$. We have
     \[
     \ell_x \cup \ell_y = \{ v = [s, 0] \text{ or } [0, s] \mid \text{for some } s \in \mathbb{R} \}
     \]
Then both \([1,0]\) and \([0,1]\) lie in \(\ell_x \cup \ell_y\). But
\[ [1,0] + [0,1] = [1,1] \notin \ell_x \cup \ell_y \]
So \(\ell_x \cup \ell_y\) is not a subspace.

6. Show that a set of vectors which contains a linearly dependent set of vectors is itself a linearly dependent set of vectors.

- Let \(S = \{v_1, \ldots, v_k\}\) be a linear dependent set of vectors and let \(T = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}\) be another set of vectors containing \(S\). Because \(S\) is a linearly dependent set, there must be a dependence relation
  \[ \alpha_1 v_1 + \cdots + \alpha_k v_k = 0_V \]
with not all \(\alpha_k = 0_V\). But then
  \[ \alpha_1 v_1 + \cdots + \alpha_k v_k + 0_F \cdot v_{k+1} + 0_F \cdot v_{k+2} + \cdots + 0_F \cdot v_m = 0_V \]
and because we know at least one of the \(\alpha_i\), \(1 \leq i \leq k\), is not equal to \(0_F\) this provides a dependence relation for \(T\). So the set \(T\) is a linearly dependent set as well.

7. Let \(\{v_1, \ldots, v_n\}\) be a basis for a (non-trivial) vector space \(V\). Show that \(v_i \neq 0_V\) for all \(i = 1, \ldots, n\).

- Suppose \(v_i = 0_V\). Then
  \[ 0_F \cdot v_1 + 0_F \cdot v_2 + \cdots + 0_F \cdot v_{i-1} + 1_F \cdot v_i + 0_F \cdot v_{i+1} + \cdots + 0_F \cdot v_n = v_i + 0_F \cdot v_1 + \cdots + 0_F \cdot v_{i-1} + 0_F \cdot v_{i+1} + \cdots + 0_F \cdot v_n = 0_V \]
Since the coefficient of \(v_i\) on the extreme left is non-zero, this identity furnishes us with a dependence relation for \(\{v_1, \ldots, v_i, \ldots, v_n\}\). Hence this set of vector is linearly dependent. But the vectors in a basis must be linearly independent. Thus, \(\{v_1, \ldots, v_i, \ldots, v_n\}\) cannot be a basis.

8. Let \(\{v_1, \ldots, v_k\}\) be a linearly independent set of vectors. Let
  \[ u = \alpha_1 v_1 + \cdots + \alpha_k v_k \]
  \[ w = \beta_1 v_1 + \cdots + \beta_k v_k \]
Prove that \(u = w\) if and only if \(\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k\).

- Suppose \(u = w\). Then \(u - w = 0_V\). But then we have
  \[ (*) \quad 0_V = u - w = (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_k - \beta_k) v_k \]
Now if any of the coefficients \(\alpha_i - \beta_i\) on the right are non-zero, then \((*)\) will furnish us with a dependence relation for the set \(\{v_1, \ldots, v_k\}\). But by hypothesis, the vectors \(\{v_1, \ldots, v_k\}\) are linearly independent – and so we’ll have a contradiction unless each \(\alpha_i - \beta_i = 0_F\). But this just means we must take \(\alpha_i = \beta_i\) for each \(i, 1 \leq i \leq k\).

\[ \iff \quad \text{Suppose } \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k. \text{ Then we have} \]
  \[ u = \alpha_1 v_1 + \cdots + \alpha_k v_k \]
  \[ = \beta_1 v_1 + \cdots + \beta_k v_k \]
  \[ = w \]

\(\Box\)