Math 4063-5023

SOLUTIONS TO FIRST EXAM

9:00 – 10:15 am, September 29, 2015

1. Definitions. Write down the precise definitions of the following notions. (5 pts each)

(a) a *subspace*

- A subset S of a vector space V over a field $\mathbb{F}$ is a *subspace* if S is closed under both scalar multiplication and vector addition; i.e.,

$$\begin{aligned} \mathbf{s} &\in S \quad , \quad \lambda \in \mathbb{F} \quad \Rightarrow \quad \lambda\mathbf{s} \in S \\ \mathbf{s},\mathbf{s}' &\in S \qquad \Rightarrow \quad \mathbf{s}+\mathbf{s}' \in S \end{aligned}$$

(b) a *dependence relation.*

- A *dependence relation* amongst a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an valid equation of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}_V$$

where the coefficients $a_1, a_2, \ldots, a_k$ are elements of the underlying field with at least one $a_i \neq 0_{\mathbb{F}}$.

(c) a *linearly independent set of vectors*

- A set of vectors is *linearly independent* if there are no dependence relations amongst the vectors.

(d) a *basis for a vector space*

- A basis for a vector space V is a set of linearly independent vectors spanning V.

(e) an $n \times m$ *homogeneous linear system*

- A *homogeneous linear system* is a set of n linear equations in m unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0_{\mathbb{F}} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0_{\mathbb{F}} \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0_{\mathbb{F}} \end{aligned}$$

2. (10 pts) Let $\{v_1, \ldots, v_k\}$ be a linearly independent set of vectors. Show that for any $w \in span(v_1, \ldots, v_k)$ there is exactly one way of expressing w as a linear combination of the vectors $v_1, \ldots, v_k$.

- Let $w \in span\,(v_1, \ldots, v_k)$ and that it has more than one expression as a linear combination of the vectors $v_1, \ldots, v_k$:

$$\begin{aligned} w &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \quad , \quad \alpha_1, \ldots, \alpha_k \in \mathbb{F} \\ w &= \beta_1 b_1 + \beta_2 v_2 + \cdots + \beta_k v_k \quad , \quad \beta_1, \ldots, \beta_k \in \mathbb{F} \end{aligned}$$

Subtracting the second equation from the first yields

(*) $$\mathbf{0}_V = w - w = (\alpha_1 - \beta_1)\, v_1 + (\alpha_2 - \beta_2)\, v_2 + \cdots + (\alpha_k - \beta_k)\, v_k$$

Since the vectors $\{v_1, \ldots, v_k\}$ are, by hypothesis, linearly independent, the only way to satisfy (*), we must have each coefficient $\alpha_i - \beta_i$ on the right hand side equal to 0_F. But then, for each $i \in 1, \ldots, k$,

$$\alpha_i - \beta_i = 0_{\mathbb{F}} \quad \Rightarrow \quad \alpha_i = \beta_i$$

Thus, the two expression for w as a linear combination of the vectors $v_1, \ldots, v_k$ must be identical.

3. Suppose V is a finitely generated vector space and S is a subspace of V.

(a) (15 pts) Prove that S is finitely generated.

(b) (5 pts) Construct a basis for S (this may be mostly done in part (a)).

(c) (10 pts) Show that if $dim(V) = dim(S)$ then $S = V$.

(a) If $S = \{0_V\}$ then S is generated by 0_V are there is nothing to prove. If $S \neq \{0_V\}$, then it contains a non-zero vector, say s_1. We have two possibilities, either $S = span\,(s_1)$ or $S \neq span\,(s_1)$. In the former case we are done; s_1 generates S.

In the latter case, there must be a non-zero vector in S lying outside the span of s_1. Let s_2 be such a vector. Note that Theorem 2 tells us that $\{s_1, s_2\}$ are linearly indepdendent. Again we have two possibilities; either $S = span\,(s_1, s_2)$ or there is a vector $s_3 \in S$ lying outside of $span\,(s_1, s_2)$ and in this case $\{s_1, s_2, s_3\}$ will be linearly independent vectors in S.

The situation bifurcates again. If $S \neq span\,(s_1, s_2, s_3)$ then there will be a fourth vector $s_4 \in S$ and $\{s_1, s_2, s_3, s_4\}$ will be a linearly independent set of vectors in S.

This process only terminates when we reach a set of linearly independent vectors $\{s_1, \ldots, s_k\}$ such that $S = span\,(s_1, \ldots, s_k)$. However, in V is n-dimensional (since V is finitely generated it will have a finite basis by Theorems 1 and 4), the cardinality of a set of linearly independent vectors in V has to be $\leq n$. Since each s_i constructed above lies in V, it is clear that the process must terminate before k exceeds n. Thus, for some $k \leq n$, we must have $S = span\,(s_1, \ldots, s_k)$ with the vectors $s_1, \ldots, s_k$ constructed as above. Thus, S is finitely generated.

(b) As remarked in part (a), the subsets $\{s_1, s_2, \ldots\}$ constructed in part (a) are sets of linearly independent vectors (by virtue of Theorem 2). The terminal set $\{s_1, \ldots s_k\}$ will be a linearly independent set of vectors spanning S. I.e., $\{s_1, \ldots, s_k\}$ will be a basis for S.

(c) The dimension of $S = span\,(s_1, \ldots s_k)$ will be k. Suppose $k = n = \dim(V)$. I claim there is no vector in V that lies outside of S. Indeed, suppose

$$v \notin S = span\,(s_1, \ldots, s_n)$$

Then, by Theorem 2, $\{s_1, \ldots, s_n, v\}$ will be a set of linearly independent vectors in V. But V is generated by n vectors, so by Theorem 1, we cannot have a set of $n + 1$ linearly independent vectors in V. Thus, no such v can exists and $S = V$.

4. Let $\mathcal{P}_{\leq 2}$ be the real vector space consisting of polynomials of degree ≤ 2:

$$\mathcal{P}_{\leq 2} \equiv \left\{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \right\}$$

(a) (10 pts) Show that $\left\{1, x, x^2\right\}$ is a basis for $\mathcal{P}_{\leq 2}$.

(b) (10 pts) Use the result of (a) to show that $\left\{1, x-1, (x-1)^2\right\}$ is also a basis for $\mathcal{P}_{\leq 2}$.

(a) We need to show that the polynomials $\left\{1, x, x^2\right\}$ is a set of linearly independent generators of $\mathcal{P}_{\leq 2}$. That $\left\{1, x, x^2\right\}$ generates $P_{\leq 2}$ is clear since every polynomial of degree ≤ 2 is a linear combination of 1, x and x^2. Suppose we had a linear combination of $1, x, x^2$ that summed to the zero polynomial

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

Then necessarily, $a_0 = 0$, $a_1 = 0$, and $a_2 = 0$. Thus $\left\{1, x, x^2\right\}$ is a set of linearly independent generators of $\mathcal{P}_{\leq 2}$ and we are done.

(b) Let us apply Theorem 6. Using the basis $\left\{1, x, x^2\right\}$, we can map polynomials in $\mathcal{P}_{\leq 2}$ to elements of $\mathcal{P}_{\leq 2}$:

$$i_B : a_0 + a_1 x + a_2 x^2 \quad \longmapsto \quad [a_0, a_1, a_2]$$

We have in particular,

$$\begin{array}{rcl} 1 & \longmapsto & [1,0,0] \\ x-1 & \longmapsto & [-1,1,0] \\ (x-1)^2 & = & 1 - 2x + x^2 \longmapsto [1,-2,1] \end{array}$$

Let us arrange the vectors on the right as the rows of a matrix

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{array} \right)$$

This matrix is easily row reduced to the row echelon form

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

Note that there are no zeros. Therefore the original three polynomials $1, x-1, (x-1)^2$ must have been linearly independent. But any set of 3 linearly independent vectors in a 3-dimensional space will be a basis for that space. Thus, $\left\{1, x-1, (x-1)^2\right\}$ is a basis for $\mathcal{P}_{\leq 2}$.

5. (15 pts) Find a basis for the $\mathbb{R}$-span of the following set of polynomials.

$$\left\{1 + 2x^2 + 3x^3 \ , \ 1 + x + 4x^2 + 4x^3 \ , \ 1 + 3x^2 + 4x^3 \ , \ 1 - x - x^2 + x^3\right\}$$

- We will apply Theorem 6, as in Problem 4; this time using the basis $\left\{1, x, x^2, x^3\right\}$.

$$\begin{array}{rcl} 1 + 2x^2 + 3x^3 & \longmapsto & [1,0,2,3] \\ 1 + x + 4x^2 + 4x^3 & \longmapsto & [1,1,4,4] \\ 1 + 3x^2 + 4x^3 & \longmapsto & [1,0,3,4] \\ 1 - x - x^2 + x^3 & \longmapsto & [1,-1,-1,1] \end{array}$$

Row-reducing the corresponding coefficient matrix, we find

$$\rightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 1 & 1 & 4 & 4 \\ 1 & 0 & 3 & 4 \\ 1 & -1 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The non-zero row vectors in the final row echelon form are $[1,0,2,3]$, $[0,1,2,1]$, $[0,0,1,1]$. Converting these vectors (in $\mathbb{R}^4$) back to polynomials provides us with a basis for the span of the original set of polynomials. Thus,

$$\left\{1+2x^2+x^3\ ,\ x+2x^2+x^3\ ,\ x^2+x^3\right\}$$

will be the desired basis.

Basic Theorems

The statements listed below you can cite and use without proof in your solutions.

Theorem 1. *Let S be a subspace of a vector space V over a field $\mathbb{F}$. Suppose S is generated by n vectors $v_1, \ldots, v_n$. Let $\{w_1, \ldots, w_m\}$ be a set of m vectors in S with $m > n$. Then the vectors $\{w_1, \ldots, w_m\}$ are linearly dependent.*

Theorem 2. *Let V be a vector space over a field $\mathbb{F}$ and let $\{v_1, \ldots, v_k\}$ be a set of linearly independent in V. Suppose $w \in V$ but $w \notin span(v_1, \ldots, v_k)$, then $\{v_1, \ldots, v_k, w\}$ is a linearly independent set of vectors in V.*

Theorem 3. *Let $V = span_{\mathbb{F}}(v_1, \ldots, v_m)$ be a finitely generated vector space. Then a basis for V can be selected from among the set of generators $\{v_1, \ldots, v_m\}$. In other words, any set of generators for a finitely generated vector space V contains a basis for V.*

Theorem 4. *Every finitely generated vector space has a basis.*

Theorem 5. *Consider a $n \times m$ linear system with coefficient matrix $\mathbf{A}$ and inhomogenous part $\mathbf{b} \in \mathbb{F}^n$. For each i between 1 and n, let $\mathbf{c}_i$ denote the element of $\mathbb{F}^n$ formed by writing the entries in the i^{th} column of $\mathbf{A}$ in order (from top to bottom). Then the linear system has a solution if and only if either of the following two conditions is satisfied.*

(i) $\mathbf{b} \in span(\mathbf{c}_1, \ldots, \mathbf{c}_m)$
(ii) $\dim span(\mathbf{c}_1, \ldots, \mathbf{c}_m) = \dim span(\mathbf{c}_1, \ldots, \mathbf{c}_m, \mathbf{b})$

Theorem 6. *Let V be an m-dimensional vector space with basis $B = [v_1, \ldots, v_m]$ and $\mathbf{A}$ be the coefficient matrix of a set of n non-zero vectors $[u_1, \ldots, u_n]$ with respect to B. Suppose that the row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n \in \mathbb{F}^m$ of $\mathbf{A}$ are in row echelon form. Then the vectors $u_1, \ldots, u_n$ are linearly independent.*

Theorem 7. *The reduced row echelon form of an $n \times m$ matrix $\mathbf{A}$ is unique.*

Theorem 8. *Let $\mathbf{A}$ be an $n \times m$ matrix, then the row space of $\mathbf{A}$ equals the dimension of its column space.*