The Isomorphism Theorems

The idea of quotient spaces developed in the last lecture is fundamental to modern mathematics. Indeed, the basic idea of quotient spaces, from a suitably abstract perspective, is just as natural and important as the notion of a subspace. Below we give the three theorems, variations of which are foundational to group theory and ring theory. (A vector space can be viewed as an abelian group under vector addition, and a vector space is also special case of a ring module.)

**Theorem 14.1 (First Isomorphism Theorem).** Let \( \phi : V \to W \) be a homomorphism between two vector spaces over a field \( \mathbb{F} \).

(i) The kernel of \( \phi \) is a subspace of \( V \).

(ii) The image of \( \phi \) is a subspace of \( W \).

(iii) The image of \( \phi \) is isomorphic to the quotient space \( V/\ker(\phi) \).

**Proof.** We have proved (i) and (ii) early on in our initial discussion of linear transformations between vector spaces. If \( V \) is finitely generated (iii) is pretty simple. Let \( B = \{b_1, \ldots, b_n\} \). Then, by Theorem 11.6 in Lecture 11, we have

\[
\text{Im}(\phi) = \text{span}(\phi(b_1), \ldots, \phi(b_n))
\]

and by Corollary 11.7 in the same lecture

\[
\dim V = \dim \text{Im}(\phi) + \dim \ker(\phi)
\]

On the other hand, at the end of the preceding lecture we had the result that if \( S \) is a subspace of \( V \) then

\[
\dim V = \dim (S) + \dim (V/S)
\]

Using \( \ker(\phi) \) for \( S \), we conclude that

\[
\dim (V/\ker(T)) = \dim (\text{Im}\phi)
\]

Theorem 12.6 says that if two finite-dimensional vector spaces have the same dimension then they are isomorphic. Therefore,

\[
V/\ker(T) \cong \text{Im}(\phi)
\]

However, the statement of Theorem 14.1 is true even when \( V \) and \( W \) are infinite dimensional. Again, since the proofs of (i) and (ii) used only the defining properties of vector space homomorphisms, we have effectively already demonstrated the validity of (i) and (ii) in the infinite-dimensional setting.

To prove (iii) in the situation where both the domain \( V \) and the codomain \( W \) might be infinite-dimensional, we'll display the isomorphism between \( V/\ker(\phi) \) and \( \text{Im}(\phi) \) explicitly.

Before proving (iii), let me first establish an important *universality property* of a canonical projection \( p_S : V \longrightarrow V/S \). This property deals with the situation where one has both a linear transformation \( \phi : V \to W \) and a particular subspace of \( V \). We thus have two linear transformations

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
p_S \downarrow & & \downarrow \\
V/S & & \end{array}
\]
THEOREM 14.2. \( \phi : V \to W \) is a linear transformation and that \( S \) is a subspace of a vector space \( V \) contained in the kernel of \( \phi \). Then there is a unique linear transformation \( \tau : V/S \to W \) with the property that

\[
\tau \circ p_S = \phi.
\]

Moreover,

\[
\ker(\tau) = \ker(\phi) / S
\]

and

\[
\im(\tau) = \im(\phi)
\]

Proof. The stipulation (i) amounts to the condition that the following diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
p_S \downarrow & & \nearrow \tau \\
V/S & & 
\end{array}
\]

commutes; in other words,

\[
\phi(v) = \tau(p_S(v)) = \tau(v + S)
\]

Now the function \( \tau \) would be well defined on \( V/S \) if and only if

\[
(v + S = u + S) \implies \tau(v + S) = \tau(u + S)
\]

But now, on the one hand, (a) is equivalent to each of the following statements

\[
\begin{align*}
& v + S = u + S \implies \tau(v + S) = \tau(u + S) \implies \phi(v) = \phi(u) \\
& v + S = u + S \implies v - u \in S \\
& s \in S \implies \phi(s) = 0_W \\
& S \subseteq \ker(\phi)
\end{align*}
\]

Thus, \( \tau : V/S \to W \) is well-defined.

Note also that

\[
\im(\tau) = \{ \tau(v + S) \mid v \in V \} = \{ \phi(v) \mid v \in V \} = \im(\phi)
\]

and

\[
\ker(\tau) = \{ v + S \mid \tau(v + S) = 0_W \}
\]

\[
= \{ v + S \mid \phi(v) = 0 \}
\]

\[
= \{ v + S \mid v \in \ker(\phi) \}
\]

\[
= \ker(\phi) / S
\]

THEOREM 14.3 (Second Isomorphism Theorem). Let \( V \) be a vector space and let \( S \) and \( T \) be subspaces of \( V \). Then

\[
\begin{align*}
& (i) \quad S + T = \{ v \in V \mid v = s + t, \ s \in S \text{ and } t \in T \} \text{ is a subspace of } V. \\
& (ii) \quad \text{The intersection } S \cap T \text{ is a submodule of } S. \\
& (iii) \quad \text{The quotient modules } (S + T)/T \text{ and } S/ (S \cap T) \text{ are isomorphic.}
\end{align*}
\]

THEOREM 14.4 (Third Isomorphism Theorem). Let \( V \) be a vector space and let \( S \) and \( T \) be subspaces of \( V \) with \( T \subseteq S \subseteq V \). Then

\[
\begin{align*}
& (i) \quad \text{The quotient space } S/T \text{ is a submodule of the quotient } V/T. \\
& (ii) \quad \text{The quotient } (V/T)/ (S/T) \text{ is isomorphic to } V/S.
\end{align*}
\]

Now we’ve already proved the first two statements of Theorem 14.1.