LECTURE 14

The Isomorphism Theorems

The idea of quotient spaces developed in the last lecture is fundamental to modern mathematics. Indeed, the basic idea of quotient spaces, from a suitably abstract perspective, is just as natural and important as the notion of a subspace. Below we give the three theorems, variations of which are foundational to group theory and ring theory. (A vector space can be viewed as an abelian group under vector addition, and a vector space is also special case of a ring module.)

Theorem 14.1 (First Isomorphism Theorem). *Let $\phi : V \to W$ be a homomorphism between two vector spaces over a field $\mathbb{F}$.*

(i) *The kernel of ϕ is a subspace of V.*
(ii) *The image of ϕ is a subspace of W.*
(iii) *The image of ϕ is isomorphic to the quotient space $V/\ker(\phi)$.*

Proof. We have proved (i) and (ii) early on in our initial discussion of linear transformations between vector spaces. If V is finitely generated (iii) is pretty simple. Let $B = \{b_1, \dots, b_n\}$. Then, by Theorem 11.6 in Lecture 11, we have

$$\mathrm{Im}(\phi) = span(\phi(b_1), \dots, \phi(b_n))$$

and by Corollary 11.7 in the same lecture

$$\dim V = \dim \mathrm{Im}(\phi) + \dim \ker(\phi)$$

On the other hand, at the end of the preceding lecture we had the result that if S is a subspace of V then

$$\dim V = \dim(S) + \dim(V/S)$$

Using $\ker(\phi)$ for S, we conclude that

$$\dim(V/\ker(T)) = \dim(\mathrm{Im}\,\phi)$$

Theorem 12.6 says that if two finite-dimensional vector spaces have the same dimension then they are isomorphic. Therefore,

$$V/\ker(T) \cong \mathrm{Im}(\phi)$$

However, the statement of Theorem 14.1 is true even when V and W are infiinite dimensional. Again, since the proofs of (i) and (ii) used only the defining properties of vector space homomorphisms, we have effectively already demonstrated the validity of (i) and (ii) in the infinite-dimensional setting.

To prove (iii) in the situation where both the domain V and the codomain W might be infinite-dimensional, we'll display the isomorphism between $V/\ker(\phi)$ and $\mathrm{Im}(\phi)$ explicitly.

Before proving (iii), let me first establish an important *universality property* of a canonical projection $p_S : V \longmapsto V/S$. This property deals with the situation where one has both a linear transformation $\phi : V \to W$ and a particular subspace of V. We thus have two linear transformations

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ p_S \downarrow & & \\ V/S & & \end{array}$$

THEOREM 14.2. *$\phi : V \to W$ is a linear transformation and that S is a subspace of a vector space V contained in the kernel of ϕ. Then there is a unique linear transformation $\tau : V/S \to W$ with the property that*

(i)
$$\tau \circ p_S = \phi \quad .$$

Moreover,

(ii)
$$\ker(\tau) = \ker(\phi)/S$$

and

(iii)
$$im(\tau) = im(\phi)$$

Proof. The stipulation (i) amounts the condition that the following diagram

$$\begin{array}{ccc} V & \overset{\phi}{\longrightarrow} & W \\ p_S \downarrow & \nearrow \tau & \\ V/S & & \end{array}$$

commutes; in other words,

$$\phi(v) = \tau(p_S(v)) = \tau(v+S)$$

Now the function τ would be well defined on V/S if and only if

(a)
$$v + S = u + S \quad \Rightarrow \quad \tau(v+S) = \tau(u+S)$$

(because to compute the value of $\tau(v+S)$, we first need to choose a representative v of $v+S$ in V). But now, on the one hand, (a) is equivalent to each of the following statements

$$\begin{aligned} v + S &= u + S \quad \Rightarrow \quad \tau(v+S) = \tau(u+S) \\ &\Longleftrightarrow \quad \phi(v) = \phi(u) \\ &\Longleftrightarrow \quad \phi(v-u) = \mathbf{0}_W \end{aligned}$$

So suppose

$$v + S = u + S$$

Then, since by hypothesis $S \subset \ker(\phi)$,

$$\begin{aligned} v - u \in S \quad &\Rightarrow \quad \phi(v-u) = \mathbf{0}_W \\ &\Rightarrow \quad \phi(v) = \phi(u) \\ &\Rightarrow \quad \tau(v+S) = \tau(u+S) \end{aligned}$$

□

Note also that

$$im(\tau) = \{\tau(v+S) \mid v \in V\} = \{\phi(v) \mid v \in V\} = im(\phi)$$

and

$$\begin{aligned} \ker(\tau) &= \{v + S \mid \tau(v+S) = \mathbf{0}_W\} \\ &= \{v + S \mid \phi(v) = 0\} \\ &= \{v + S \mid v \in \ker\phi\} \\ &= \ker(\phi)/S \end{aligned}$$

The preceding theorem now enables a simple proof of statement (iii) of the First Isomorphism Theorem. Let $\phi : V \to W$ be a vector space homomorphism and choose $S = \ker(\phi)$. Then by the preceding theorem we have an induced homomorphism

$$\tau : V/\ker\phi \to W$$

such that

$$\begin{aligned} im(\tau) &= im(\phi) \\ \ker(\tau) &= \ker(\phi)/\ker(\phi) = \{\mathbf{0}\} \end{aligned}$$

In other words, τ is a surjective homomorphism from $V/\ker\phi$ onto $im(\phi)$ that has has kernel $\{\mathbf{0}\}$. τ is thus an isomorphism and Theorem 14.1 (iii) is established. □

There are two other isomorphism theorems

THEOREM 14.3 (Second Isomorphism Theorem). *Let V be a vector space and let S and T be subspaces of V. Then*

(i) $S+T = \{v \in V \mid v = s+t \quad , \quad s \in S \text{ and } t \in T\}$ *is a subspace of V.*
(ii) *The intersection $S \cap T$ is a submodule of S.*
(iii) *The quotient modules $(S+T)/T$ and $S/(S\cap T)$ are isomorphic.*

Proof.

(i) Let $w = s+t$, $w' = s'+t'$ be generic elements of $S+T$ and let $\alpha, \beta \in \mathbb{F}$. Then

$$\alpha w + \beta w' = \alpha(s+t) + \beta(s'+t') = (\alpha s + \alpha s') + (\beta t + \beta t') \in S+T$$

So $S+T$ is closed under linear combinations, and hence $S+T$ is a subspace of V.

(ii) Let $w, w' \in S\cap T$. Then $w, w' \in S$ and $w, w' \in T$ and

$$\begin{aligned} w, w' &\in S \quad \text{and} \quad w, w' \in T \qquad \Rightarrow \qquad \alpha w + \beta w' \in S \quad \text{and} \quad \alpha w + \beta w' \in T \\ &\Rightarrow \qquad \alpha w + \beta w' \in S \cap T \end{aligned}$$

and so $S \cap T$ is a submodule of S (and is also a submodule of T and a submodule of V)

(iii) Let $i : S \to S+T$, be the natural embedding

$$i(s) = s + 0_V \in S+T$$

It is easy to check this is a linear transformation. When we compose it with the canonical projection $p : S+T \to (S+T)/T$, we obtain a linear transformation $f = p \circ i : S \to (S+T)/T$.

$$S \xrightarrow{\ i\ } S+T \xrightarrow{\ p\ } (S+T)/T \quad \Rightarrow \quad f = p\circ i : S \to (S+T)/T$$

The kernel of f is going to consist of those elements of S that get sent by i to the kernel of p; in other words,

$$\begin{aligned} \ker(f) &= \{s \in S \mid i(s) \in \ker(p)\} \\ &= \{s \in S \mid s + 0_V \in T\} \\ &= \{s \in S \mid s \in T\} \\ &= S \cap T \end{aligned}$$

So

$$\ker(f) = S \cap T$$

I claim that

$$Range(f) = (S+T)/T$$

Indeed, suppose $w \in (S+T)/T$, then since p_T is surjective, $w = p(s+t)$ for some $s+t \in S+T$. But then

$$\begin{aligned} w &= p(s+t) \\ &= p(s) + p(t) \\ &= p(s) \\ &= p(s + 0_V) \\ &= p(i(s)) \\ &= f(s) \end{aligned}$$

Thus, every element of $(S+T)/T$ is in the range of f.

Having displayed $f : S \to (S+T)/T$ as a surjective linear transformation, the First Isomorphism Theorem now implies

$$(S+T)/T = Range(f) \quad \approx \quad S/\ker(f) = S/(S \cap T)$$

□

Theorem 14.4 (Third Isomorphism Theorem). *Let V be a vector space and let S and T be subspaces of V with $T \subseteq S \subseteq V$. Then*

(i) *The quotient space S/T is a submodules of the quotient V/T.*
(ii) *The quotient $(V/T)/(S/T)$ is isomorphic to V/S.*

Proof.

(i) This is more-or-less self evident from the defintions. Each element $s + T \in S/T$ can be identified with the element $s + T \in V/T$ since $s \in V$ (as S is a subspace of V). Thus, S/T is naturally a subset of V/T. It is S/T is also closed under linear combinations (with the rules for addition and scalar multiplication it inheritsfrom V), and so it is, in fact, a subspace of V.

(ii) Let's just construct the space $(V/T)/(S/T)$. An element of V/T is a hyperplane of form

$$v + T \equiv \{v + t \mid t \in T\}$$

An element of $(V/T)/(S/T)$ is a hyperplane of hyperplanes

$$\begin{aligned} (v+T) + S/T &= \{(v+t_1) + (s+t_2) \mid s \in S \ , \ t_1, t_2 \in T\} \\ &= \{v + s + t_1 + t_2 \mid s \in S \ , \ t_1, t_2 \in T\} \end{aligned}$$

However, since $t_1, t_2 \in T \subset S$, we have $s + t_1 + t_2 \in S$ and so we have

$$(v+T) + S/T = \{v + s' \mid s' \in S\ \} \in V/S$$

Thus, the two quotient spaces $(V/T)/(S/T)$ and V/S identify with exactly the same set of hyperplanes in V. That shows that $(V/T)/(S/T)$ and V/S coincide as sets; however, a vector space is a set with additional structures; namely, scalar multiplication and vector addition. So to identify $(V/T)/(S/T)$ with V/S as vector spaces, we need to check that their respective rules for scalar multiplication and vector addition coincide as well. But this, in fact, will be the case, because in both quotient spaces the rules for scalar multiplication and vector are inherited from those of V. □