LECTURE 9

Hyperplanes

Let $V$ be a finitely generated vector space over a field $F$. Today we will focus our attention on a special class of subsets of $V$. These subsets will not in general be subspaces, but they arise quite naturally in linear algebra and have a lot of nice properties.

**Definition 9.1.** Let $V$ be a vector space over a field $F$, let $S$ be a subspace of $V$ and let $p \in V$. Let

$$M_{p, S} := \{ p + s \mid s \in S \}$$

We shall refer to such a subset as a **linear submanifold** of $V$. The subspace $S$ is called the **directing subspace** of $M_{p, S}$. The **dimension** of $M_{p, S}$ is defined as the dimension of $S$.

**Remark 9.2.** The notion of a manifold is actually more germane to differential geometry than linear algebra, in the geometric setting the notion of a linear manifold is akin to the notion of a linear function in Calculus - it's so simple that it's not worth discussing except as a simplifying limit.

However, because we are not discussing differential geometry in this course, I don’t see much point in mentioning manifolds. What I think would be more helpful will be to view sets of the form $M_{p, S}$ as generalizations of lines and planes in $\mathbb{R}^3$. Indeed, you can generate a line in $\mathbb{R}^3$ by starting at a particular point $p \in \mathbb{R}$ and then heading off an arbitrary distance $d$ (forwards and backwards) along a particular direction $v$: that is to say

$$\text{line} = \text{set of the form } \{ p + tv \mid p, v \in \mathbb{R}^3, t \in \mathbb{R} \}$$

Since the vectors $\{tv \mid t \in \mathbb{R} \}$ constitute the span of $v$, such a line is a linear submanifold of $\mathbb{R}^3$ as defined above. Similarly, a plane in $\mathbb{R}^3$ is formed by starting at a particular point and then heading an arbitrary distance in two possible directions; i.e. a subset of $\mathbb{R}^3$ of the form

$$\text{plane} = \{ p + su + tv \mid s, t \in \mathbb{R} \}$$

$$= \{ p + s \mid s \in \text{span} (u, v) \}$$

To underscore this simple geometric picture, I shall henceforth refer to linear submanifolds of a vector space $V$ as a **hyperplane** in $V$.

**Definition 9.3.**

**Notation 9.4 (common but abusive notation).** Let $V$ be a vector space. If $S$ is a subspace of a vector space $V$, and $b$ is a point of $V$ we shall write $b + S$ for the corresponding hyperplane in $V$. (What’s abusive about this is that you can’t really add a subspace to a vector; on the other hand, if you interpret this expression as adding every vector in $S$ to $b$ then it does kind of make sense.)

I note also that we have already run into hyperplanes in two particular contexts. In Lecture 6, I defined quotient spaces $V/S$, $S$ being some subspace of a vector space $V$, as the collection of sets of the form $p_0 + S$ ($= [p_0]_S$ in the notation of Lecture 6).

In Theorem 7.7 of Lecture 7, we saw hyperplanes arise as the solution sets of linear systems.
Theorem 9.2. (rephrased) Suppose $Ax = b$ is an $n \times m$ linear system. Then the solution set of this linear system can be expressed as

$$ p_0 + S = \{ p_0 + s \mid s \in S \} $$

where $p_0$ is any particular solution of $Ax = b$ and $S$ is the solution set of $Ax = 0$.

The first thing to point out about hyperplanes is that in general they are not subspaces. Here is a simple counter-example. Let $V = \mathbb{R}^2$, $p_0 = [1,0]$ and let $S = \text{span}([0,1])$. Then

$$ p_0 + S = \{ [1, y] \mid y \in \mathbb{R} \} $$

which cannot be a subspace of $\mathbb{R}^2$ since it does not contain the zero vector in $\mathbb{R}^2$.

Theorem 9.2 told us how to view the solution set of homogeneous linear system $Ax = 0$. The following lemma provides a converse to this result.

**Theorem 9.5.** Let $b + S$ be a hyperplane in a vector space $V$. Then

$$ S = \{ v \in V \mid v = r - q , \ r, q \in b + S \} $$

**Proof.** Let

$$ \tilde{S} = \{ v \in V \mid v = r - q , \ r, q \in b + S \} $$

Suppose $v \in \tilde{S}$. Then there exists $r = b + s_1$ and $q = b + s_2$ in $b + S$ such that

$$ v = r - q = (b + s_1) - (b + s_2) = s_1 - s_2 \in S $$

and so every element $v \in \tilde{S}$ is also a vector of $S$.

Suppose on the other hand that $s \in S$, then we can always write

$$ s = s + b - b = (b + s) - (b + 0) $$

which displays $s$ as an element of $\tilde{S}$. We conclude $\tilde{S} = S$ and thus prove the theorem. \qed

Theorem 9.2 told us how to view the solution set of homogeneous $n \times m$ linear system is subspace of $\mathbb{F}^m$. The following lemma provides a converse to this result.

**Lemma 9.6.** Let $S$ be an $r$-dimensional subspace of a vector space $V$ of dimension $m$. Then there exists a set of $m - r$ homogeneous linear equations in $m$ unknowns whose solution set is exactly $S$.

Let $\{b_1, \ldots, b_r\}$ be a basis for $S$. Consider the solution space $S^*$ of

$$ b_1 \cdot x = 0 $$
$$ b_2 \cdot x = 0 $$
$$ \vdots $$
$$ b_r \cdot x = 0 $$

We first note that $S^*$ is not likely to coincide with $S$, simply because for example, $b_1 \in S$ but $b_1 \cdot b_1 \neq 0$. On the other hand, since the vectors $b_i$ are all linearly independent, it follows that the coefficient matrix $A$ for this linear system has rank $r$ (since the row space of $A$ will be span of the $r$ linearly independent vectors $b_1, \ldots, b_r$). So the solution space $S^*$ of $Ax = 0$ will be of dimension $m - r$. Let $\{c_1, \ldots, c_{m-r}\}$ be the basis for the $S^*$ and consider the system

$$ c_1 \cdot x = 0 $$
$$ \vdots $$
$$ c_{m-r} \cdot x = 0 $$

Let $S^{**}$ denote the solution set of $c_1 \cdot x = 0$. Clearly, each $b_i$ will be a solution of this system, and thus so will any linear combination of the vectors $b_i$, and thus, the entire subspace $S$ lie in the solution set of $c_1 \cdot x = 0$. On the other hand, Since the vectors $c_1, \ldots, c_{m-r}$ are linearly independent, it is clear that the rank of this

\[1\] That the solution set of a homogeneous $n \times m$ linear system $Ax = 0$ is actually a subspace of $\mathbb{F}^m$ is the content of Theorem 9.2.
linear system is \( m - r \) and so solution set of dimension \( m - (m - r) = r \). But we’ve seen that if a subspace has same dimension as the vector space containing it, the subspace must be the whole vector space. Since \( S \subset S^{**} \) and \( \dim(S) = \dim(S^{**}) \) we conclude that \( S \) coincides with the solution set \( S^{**} \) of (*) \( \square \).

**Example 9.7.** Find a homgeneous linear system whose solution set coincides with the span of \([1, 0, 1] \) and \([1, 1, 0]\).

- We first find a basis for the solution set of
  \[
  0 = [1, 0, 1] \cdot x = x_1 + x_3
  \]
  \[
  0 = [1, 1, 0] \cdot x = x_1 + x_2
  \]

  The augmented matrix for this system is
  \[
  \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  1 & 1 & 0 & 0
  \end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & -1 & 0
  \end{bmatrix}
  \]

  and so the general solution will be
  \[
  x_1 = -x_3 \\
  x_2 = x_3
  \]

  or
  \[
  \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
  \end{pmatrix} = \begin{pmatrix}
  -x_3 \\
  x_3 \\
  1
  \end{pmatrix}
  \]

  Thus, the solution space has basis \( c_1 = [-1, 1, 1] \). The desired homogeneous linear system will be
  \[
  [-1, 1, 1] \cdot x = 0
  \]

**Theorem 9.8.** A necessary and sufficient condition for a subset \( M \) of vectors to form a hyperplane in \( F^n \) of dimension \( r \) is that \( M \) be the set of solutions of a system of \( m - r \) equations in \( m \) unknowns whose coefficient matrix has rank \( r \).

**Proof.** How a solution set of a linear system constitutes a hyperplane was explained in at the start of this lecture. To see that every hyperplane \( b + S \) corresponds to a linear system, we just observe that by Lemma 10.5 the directing subspace \( S \) can be viewed as the solution set an \( (m - r) \times m \) linear system \( Ax = 0 \). Let

\[
\tilde{b} = Ab
\]

Then any \( b + s \) vector in \( b + S \) will satisfy

\[
A(b + s) = Ab + 0 = \tilde{b}
\]

This shows that the solution of

\[
(*)
\]

will contain \( b + S \). On the other hand, by construction \( y = b \) is a solution of \( (*) \) and by Theorem 7.7, any other solution of \( Ay = \tilde{b} \) will be of the form

\[
b + \text{some solution of } Ax = 0
\]

and so any solution \( y \) of \( (*) \) will be of the form

\[
y = b + s , \quad s \in S
\]

Therefore \( b + S \) will coincide with the solutions of \( (*) \). \( \square \)

Finally, let me describe an algorithm by which one can identify a linear system whose solution set is a given hyperplane.

We have see above that if we had a hyperplane in \( \mathbb{R}^m \) which is also a subspace \( S \) of \( \mathbb{R}^m \), then we could construct a corresponding equation set as follows:
9. HYPERPLANES

- find a basis \{v_1, \ldots, v_k\} for \(S\)
- find a basis \{u_1, \ldots, u_\ell\} for the solution set of the linear system

\[
\begin{align*}
v_1 \cdot x &= 0 \\
v_2 \cdot x &= 0 \\
&\vdots \\
v_k \cdot x &= 0
\end{align*}
\]

- The equations that cut out the subspace \(S\) will

\[
\begin{align*}
u_1 \cdot x &= 0 \\
u_2 \cdot x &= 0 \\
&\vdots \\
u_\ell \cdot x &= 0
\end{align*}
\]

Now suppose we have a hyperplane in \(\mathbb{R}^m\) of the form

\[
H = p_0 + S \equiv \{p_0 + s \mid s \in S\}
\]

\(S\) being some subspace of \(\mathbb{R}^m\). Suppose also that we have followed the algorithm above and found \(\ell\) vectors \(u_1, \ldots, u_\ell\) such that

\[
s \in S \iff u_i \cdot s = 0
\]

Then each vector in \(H\) will satisfy

\[
u_i \cdot (p_0 + s) = u_i \cdot p_0 + u_i \cdot s = u_i \cdot p_0 + 0 = u_i \cdot p_0, \quad i = 1, \ldots, \ell
\]

And so the linear equations whose solution set is the hyperplane \(H = p_0 + S\) will be

\[
\begin{align*}
u_1 \cdot x &= u_1 \cdot p_0 \\
u_2 \cdot x &= u_2 \cdot p_0 \\
&\vdots \\
u_\ell \cdot x &= u_\ell \cdot p_0
\end{align*}
\]