LECTURE 3

Dimension and Bases

In the preceding lecture, we introduced the notion of a subspace of a vector space and an easy way to construct subspace; namely, by considering the set of all possible linear combinations of a set of vectors: if \( \{v_1, v_2, \ldots, v_k\} \) is a set of vectors, then

\[
\text{span}_F (v_1, \ldots, v_k) := \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in F \}
\]

will be a subspace. We may also refer to \( \text{span}_F (v_1, \ldots, v_k) \) as the subspace generated by \( v_1, \ldots, v_k \).

We noted that such a subspace may be generated may different ways; the exact same subspace being produced for different choices of \( v_1, \ldots, v_k \). In order to make the presentation of a vector in a subspace as simple as possible it made sense to try to work with as few generators as possible. This lead us to the following definition

**Definition 3.1.** A set of vectors is **linearly dependent** if an equation of the form

\[
\alpha_1 v_1 + \cdots + \alpha_k v_k = 0_v
\]

can be satisfied without setting all the coefficients \( \alpha_1, \ldots, \alpha_k \) equal to 0.

This definition is related to the problem of finding a minimal set of generators by

**Proposition 3.2.** \( \text{span}_F (v_1, \ldots, v_k) = \text{span}_F (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \) if and only if there is a dependence relation amongst the vectors \( v_1, \ldots, v_k \) for which the coefficient \( \alpha_i \) of \( v_1 \) is non-zero.

So anytime there is a viable dependence relation we can toss out a generator without changing the nature of the subspace generated. When we there are no more viable dependence relations, this procedure terminates. The final minimal set of generators are then **linearly independent** since dependence relations amongst the vectors no longer exist.

We now turn to the questions of how many generators do we end up with; does this number depend on the vectors that we start with, or on the choices we make in removing superfluous generators.

**Theorem 3.3.** Let \( S \) be a subspace of a vector space \( V \) over a field \( F \). Suppose \( S \) is generated by \( n \) vectors \( v_1, \ldots, v_n \). Let \( \{w_1, \ldots, w_m\} \) be a set of \( m \) vectors in \( S \) with \( m > n \). Then the vectors \( \{w_1, \ldots, w_m\} \) are linearly dependent.

**Proof.** We will do a proof by induction.

Suppose \( n = 1 \). Then each element of \( S \) is of the form \( \lambda v_1 \) for some \( \lambda \in F \). So there must be a choice of scalars \( \lambda_1, \ldots, \lambda_m \) so that

\[
w_i = \lambda_i v_1 , \quad i = 1, \ldots, m
\]

But then

\[-\lambda v_1 + w_1 = 0_v
\]

is a dependence relation amongst the \( w_1, \ldots, w_m \) and so \( \{w_1, \ldots, w_m\} \) are linearly dependent.
Now assume the statement is true for \( n = N - 1 \). Write each \( w_i, i = 1, \ldots, m > N \), as

\[
    w_i = \sum_{j=1}^{N} \alpha_{ij} v_j
\]

Case 1: all \( \alpha_{iN} = 0 \). In this situation each \( w_i \) lies in the span of the first \( N - 1 \) \( v_i \), and so the conclusion that \( \{w_1, \ldots, w_m\} \) are linearly dependent is affirmed by the induction hypothesis.

Case 2: At least one \( \alpha_{iN} \neq 0 \). By, if necessary, reordering the \( v_i \), we can assume that \( \alpha_{1N} \neq 0 \). Now consider

\[
    w_2 = \frac{\alpha_{2N}}{\alpha_{1N}} w_1
\]

The coefficient of \( v_N \) is this expression is, by construction, equal to 0 (\( \neq \)). Similarly, the vectors

\[
    w_3 - \frac{\alpha_{3N}}{\alpha_{1N}} w_1
\]

\[
    \vdots
\]

\[
    w_m = \frac{\alpha_{mN}}{\alpha_{1N}} w_1
\]

all have 0 (\( \neq \)) component along \( v_N \). Thus, the \( m - 1 \) vectors \( w_2 - \frac{\alpha_{2N}}{\alpha_{1N}} w_1, \ldots, w_m - \frac{\alpha_{mN}}{\alpha_{1N}} w_1 \), all belong to the subspace generated by \( v_1, \ldots, v_{N-1} \). Since \( m > N \) implies \( m - 1 > N - 1 \), the induction hypothesis implies that these \( m - 1 \) vectors must be linearly dependents. So there are constants \( \beta_2, \ldots, \beta_m \), not all equal to 0 such that

\[
    \beta_2 \left( w_2 - \frac{\alpha_{2N}}{\alpha_{1N}} w_1 \right) + \cdots + \beta_m \left( w_m - \frac{\alpha_{mN}}{\alpha_{1N}} w_m \right) = 0_v
\]

or

\[
    \left( \frac{\beta_2 \alpha_{2N}}{\alpha_{1N}} + \cdots + \frac{\beta_m \alpha_{mN}}{\alpha_{1N}} \right) w_1 + \beta_2 w_2 + \cdots + \beta_m w_m = 0
\]

which is a dependence relation amongst \( \{w_1, \ldots, w_m\} \) since at least one of the coefficients of the last \( m - 1 \) terms must be non-zero. \( \square \)

**Corollary 3.4.** Suppose \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_m\} \) are two sets of generators of a subspace \( S \) and that both sets of generators are linearly independent. Then \( n = m \).

**Proof.** If \( m > n \), then we will would have more vectors in the set \( \{w_1, \ldots, w_m\} \) than we have generators in set \( \{v_1, \ldots, v_n\} \). By the preceding theorem, we would conclude that the set \( \{w_1, \ldots, w_m\} \) must be a linearly dependent set of vectors. But that violates our hypothesis. If \( n > m \), the a similar argument, in which the roles of the sets \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_m\} \) are switched would force us to conclude that the vectors \( \{v_1, \ldots, v_n\} \) are linearly independent, in violation of our hypothesis. The only other possibility left is \( n = m \) and so the statement is proved. \( \square \)

Thus the number of linearly independent vectors used to generate a subspace is independent of the choice of a linearly independent set of generators. This is an important invariant of a subspace and the motivation for the following definition.\(^1\)

**Definition 3.5.** The common cardinality of any linearly independent set of generators for a subspace \( S \) is called the **dimension** of \( S \).

**Definition 3.6.** A **basis** for a subspace \( S \) is a linearly independent set of generators for \( S \).

So, the dimension of a subspace \( S \) is the number of vectors in any basis of \( S \).

\(^1\)A notion that depends ostensibly on some choices (in a present case the choice of a linearly independent set of generators), but which in fact is independent of the choices made, we sometimes refer to as an invariant of the construction.