1. Let \( \mathcal{P} \) be the vector space of polynomials with indeterminant \( x \). Which of the following mappings are linear transformations from \( \mathcal{P} \) to itself:

(a) \( T : p \rightarrow xp \)
(b) \( T : p \rightarrow 2p \)
(c) \( T : p \rightarrow \frac{dp}{dx} + 2p \)
(d) \( T : p \rightarrow \int_0^1 p(x) \, dx \)

2. Suppose \( f : V \rightarrow W \) is a linear transformation:

(a) Prove that \( f \) is injective if and only if \( \ker(f) = \{ 0_V \} \).
(b) Prove that \( f \) is surjective if and only if \( \dim(\text{Im}(f)) = \dim W \).
(c) Prove that if \( f \) is bijective if and only if \( \dim(V) = \dim(W) \) and \( \ker(f) = \{ 0_V \} \).

3. Prove that the composition \( f \circ g \) of two linear transformations is a linear transformation.

4. Consider the mapping \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) \( T([x_1, x_2]) = [x_1 - x_2, x_1 + x_2, x_1 - 2x_2] \)

(a) Show that \( T \) is a linear transformation.
(b) Find the matrix corresponding to \( T \) and the natural bases of \( B = \{ [1, 0], [0, 1] \} \) and \( B' = \{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \} \) of, respectively, \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
(c) What is the kernel of this linear transformation.
(d) What is the range (i.e. the image) of this linear transformation.

5. Let \( \mathcal{P}_3 \) be the vector space of polynomials of degree \( \leq 3 \) with natural basis \( \{ x^3, x^2, x, 1 \} \). Find the matrix \( T_{B,B} \) corresponding to the linear transformation \( T : \mathcal{P}_3 \rightarrow \mathcal{P}_3 \), \( p \rightarrow 2x \frac{dp}{dx} + p \) and the basis \( B \) (same basis for the domain and codomain of \( T \)).

6. Suppose \( f : V \rightarrow W \) is a linear transformation and \( S \) is a subspace of \( W \) contained in \( \text{Im}(f) \). Prove that \( f^{-1}(S) = \{ v \in V \mid f(v) \in S \} \) is a subspace of \( V \).

7. Let \( S \) be a subspace of a vector space \( V \) over a field \( \mathbb{F} \) and let \( V/S \) be the corresponding quotient space: \( V/S := \{ v + S \mid v \in V \} \) where \( v + S := \{ v' \in V \mid v' = v + s \text{ for some } s \in S \} \). Let addition and scalar multiplication of elements of \( V/S \) be defined by

\[ + : V/S \times V/S \rightarrow V/S ; \quad (v + S) + (w + S) := (v + w + S) \]
\[ \ast : \mathbb{F} \times V/S \rightarrow V/S ; \quad \lambda(v + S) := (\lambda v + S) \]

Show that \( V/S \) is a vectors space over \( \mathbb{F} \) (i.e., verify all 8 axioms for a vector space).

8. Let \( S \) be the subspace of \( \mathbb{R}^3 \) spanned by \( [1, 0, 0] \) and \( [0, 1, 0] \). Identify let \( v_1 = [1, -1, 3] \) and let \( v_2 = [2, 3, 1] \). Determine \( [v_1]_S + [v_2]_S \) explicitly (it has to be some hyperplane in the direction of \( S \) inside \( \mathbb{R}^3 \)).