1. Let $F$ be a field, and let $F^n$ denote the set of $n$-tuples of elements of $F$, with operations of scalar multiplication and vector addition defined by

$$
\lambda \cdot [\alpha_1, \ldots, \alpha_n] : = [\lambda \alpha_1, \ldots, \lambda \alpha_n], \quad \text{for all } \lambda \in F \text{ and all } [\alpha_1, \ldots, \alpha_n] \text{ in } F^n
$$

$$
[\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] : = [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n], \quad \text{for all } [\alpha_1, \ldots, \alpha_n] \text{ and } [\beta_1, \ldots, \beta_n] \text{ in } F^n
$$

Check that $F^n$ satisfies all the axioms of a vector space over $F$.

2. Let $C^1(\mathbb{R})$ be the set of continuous, differentiable functions on the real line with values in $\mathbb{R}$. Define scalar multiplication and vector addition on $C^1(\mathbb{R})$ by

$$
(\lambda \cdot f)(x) : = \lambda f(x), \quad \forall \lambda \in \mathbb{R}, \quad \forall f \in C^1(\mathbb{R});
$$

$$
(f + g)(x) : = f(x) + g(x), \quad \forall f, g \in C^1(\mathbb{R}).
$$

Check that $C^1(\mathbb{R})$ satisfies the axioms for a vector space over $\mathbb{R}$.

3. Determine which of the following subsets are subspaces of $C^1(\mathbb{R})$

(a) The set of polynomial functions in $C^1(\mathbb{R})$.

(b) The set of all functions $f \in C^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right)$ is a rational number.

(c) The set of all $f \in C^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right) = 0$.

(d) The set of all $f \in C^1(\mathbb{R})$ such that $\int_0^1 f(x) \, dx = 1$

(e) The set of all $f \in C^1(\mathbb{R})$ such that $\int_0^1 f(x) \, dx = 0$

(f) The set of all $f \in C^1(\mathbb{R})$ such that $\frac{df}{dx} = 0$.

4. Prove that a subspace (a subset of a vector space closed under vector addition and scalar multiplication) is a vector space by verifying all 8 axioms.

5. Is the intersection of two subspaces a subspace (prove your answer)?

6. Is the union of two subspaces a subspace (explain your answer)?

7. Show that a set of vectors which contains a linearly dependent set of vectors is itself a linearly dependent set of vectors.

8. Let $\{v_1, \ldots, v_n\}$ be a basis for a (non-trivial) vector space $V$. Show that $v_i \neq 0_V$ for all $i = 1, \ldots, n$.

9. Let $\{v_1, \ldots, v_k\}$ be a linearly independent set of vectors. Let

$$
u = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

$$w = \beta_1 v_1 + \cdots + \beta_k v_k$$

Prove that $u = w$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$.

10. Show that $\{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space $P_n$ of polynomials of degree $\leq n$. (Hint: just check that the definition of a basis is satisfied.)