LECTURE 20

Properties of Continuous Functions

Recall

DEFINITION 20.1. *Let $f : D \to \mathbb{R}$ and let $c \in D$. We say that f is* ***continuous*** *at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left.\begin{array}{c} x \in D \\ \text{and} \\ |x - c| < \delta \end{array}\right\} \quad \Rightarrow \quad |f(x) - f(c)| < \varepsilon$$

If f is continuous at each point of a subset S of D, then f is said to be ***continuous on*** *S. If f is continuous at each point of its domain D, then f is said to be a continuous function.*

The goal of this lecture is to examine the how the existence of a continuous function $f : D \to \mathbb{R}$ places restrictions on the topological nature of subsets D and $f(D)$.

EXAMPLE 20.2. Consider the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. This is function is continuous at each point of its domain. But although its domain is bounded, its range $f(D) = (0, +\infty)$ is clearly unbounded.

The point of this example is to provide contrast with the following theorem.

THEOREM 20.3. *Let D be a compact set and suppose that $f : D \to \mathbb{R}$ is a continuous. Then $f(D)$ is compact.*

Proof. By the Heine-Borel theorem it suffices to show that $f(D)$ is closed and bounded.

LEMMA 20.4. *Every bounded sequence has a convergent subsequence.*

Suppose that $f(D)$ is not bounded. Then for each $n \in \mathbb{N}$ there exists a point $x_n \in D$ such that $|f(x_n)| > n$. Since D is bounded, the Lemma implies the sequence $(x_1, x_2, \dots)$ in D will have convergent subsequence (x_{n_k}) and hence an accumulation point x_o. Since D is also closed such an accumulation point must lie in D. Therefore, f is continuous at x_0. And so by Theorem 21.2 $f(x_{n_k})$ will converge to $f(x_0)$. In particular, $(f(x_{n_k}))$ will be bounded. But this contradicts the hypothesis that

$$|f(x_{n_k})| > n_k > k \text{ for all } k \in N$$

We now show that $f(D)$ must be closed. Let (y_n) be a convergent sequence in $f(D)$ and let $y = \lim y_n$. If suffices to show that $y \in f(D)$. Since $y_n \in f(D)$ for each n, for each y_n we can choose an $x_n \in D$ such that $y_n = f(x_n)$. Since D is closed and bounded, there will exist a subsequence (x_{n_k}) of (x_n) that converges to some point x_0 in D. Since f is continuous at x_0 we'll have

$$\begin{aligned} f(c) &= \lim_{x \to c} f(x) = \lim (f(x_{n_k})) \\ &= \lim (y_{n_k}) \\ &= y \end{aligned}$$

Thus, $y = f(c) \in f(D)$ and so $f(D)$ contains its accumulation points so $f(D)$ is compact.

COROLLARY 20.5. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes a minimum and maximum values on D. That is to say, there exists points $x_1, x_2 \in D$ such that*

$$\begin{aligned} f(x_1) &\leq f(x) \quad \text{for all } x \in D \\ f(x_2) &\geq f(x) \quad \text{for all } x \in D \end{aligned}$$

Proof. Since f is continuous and D is compact, by the preceding theorem $f(D)$ is compact; hence $f(D)$ is closed. Hence $f(D)$ contains its boundary points. Hence, $f(D)$ has a minimal and a maximal element. Let

$$\begin{aligned} y_1 &= \min f(D) \\ y_2 &= \max f(D) \end{aligned}$$

Since $y_1 \in f(D)$, there exists an $x_1 \in D$ such that $y_1 = f(x_1)$. Similarly, there exists an $x_2 \in D$ such that $y_2 = f(x_2)$. Now we have

$$f(x_1) = y_1 \leq y(x) \leq y_2 = f(x_2) \quad \text{for all } x \text{ in } D$$

and the proposition is proved. ∎

LEMMA 20.6. *Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(x) = 0$.*

Proof. Set

$$S = \{x \in [a, b] \mid f(x) \leq 0\}$$

This set is non-empty since $a \in S$ and it is bounded since $x \in S \quad \Rightarrow |x| \leq \max\{|a|, |b|\}$. Thus, by the Completeness Property of the Reals, S has a least upper bound. Set

$$c = \sup(S)$$

I claim $f(c) = 0$.

- Suppose $f(c) < 0$. Then $V = N\left(f(c), \frac{f(c)}{2}\right)$ will be a neighborhood of $f(c)$ such that every element of V is less than zero. Since f is continuous, to V there must correspond a neighborhood $U = N(c, \varepsilon)$ of c such that
$$x \in U \quad \Rightarrow \quad f(x) \in V$$
(Theorem 21.2). But then
$$f(c + \frac{\varepsilon}{2}) < 0$$
and so
$$c + \frac{\varepsilon}{2} \in S$$
and so c is not an upper bound of set S.
- Suppose $f(c) > 0$. Then $W = N\left(f(c), \frac{f(c)}{2}\right)$ will be a neighborhood of $f(c)$ such that every element of V is greater zero. Since f is continuous, to W there must correspond a neighborhood $U' = N(c, \varepsilon')$ of c such that
$$x \in U' \quad \Rightarrow \quad f(x) \in W$$
In particular, $x = c - \frac{\varepsilon'}{2} \in U'$ and so for all x between $c - \frac{\varepsilon}{2}$ and c
$$f(x) > 0 \quad \Rightarrow \quad x \notin S$$
and so c can not be the **least** upper bound for S.
- We conclude that if $c = \sup(S)$ then $f(c) = 0$. Since $\sup(S)$ is guaranteed to exist, we are done.

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THEOREM 20.7 (Intermediate Value Theorem). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then if k is any real number between $f(a)$ and $f(b)$, there exists a point $c \in [a, b]$ such that $f(c) = k$*

Proof. Suppose
$$f(a) < k < f(b)$$
Consider the function $g : [a, b] \to \mathbb{R}$ defined by
$$g(x) = f(x) - k.$$
Note that g is continuous since f is continuous. We also have
$$g(a) < 0 < g(b)$$
and so, by the preceding lemma, there exists a $c \in [a, b]$ such that $g(c) = 0$. But then this implies
$$f(c) = k.$$
Suppose
$$f(b) < k < f(a)$$
Consider the function $h : [a, b] \to \mathbb{R}$ defined by
$$h(x) = k - f(x).$$
Note that h is continuous since f is continuous. We also have
$$h(b) < 0 < h(a)$$
and so, by the preceding lemma, there exists a $c \in [a, b]$ such that $h(c) = 0$. But then this implies
$$f(c) = k.$$
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