LECTURE 19

Continuous Functions

Definition 19.1. *Let $f : D \to \mathbb{R}$ and let $c \in D$. We say that f is* ***continuous*** *at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left.\begin{array}{c} x \in D \\ \text{and} \\ |x-c| < \delta \end{array}\right\} \quad \Rightarrow \quad |f(x) - f(c)| < \varepsilon$$

If f is continuous at each point of a subset S of D, then f is said to be ***continuous on*** *S. If f is continuous at each point of its domain D, then f is said to be a continuous function.*

Remark 19.2. Note that the point c need not be an acccumulation point. And so continuity of f at c is not quite the same as saying

$$\lim_{x \to c} f(x) = f(c)$$

(simply because the definition of the left hand side requires that c be an accumulation point). On the other hand, if $c \in D$ but c is not an accumulation point of D, then a function $f : D \to \mathbb{R}$ will always be continuous at c. Because in this case, c will be an isolated point of D and so it will always be possible to find a $\delta > 0$ such that

$$\left.\begin{array}{c} x \in D \\ \text{and} \\ |x-c| < \delta \end{array}\right\} \quad \Rightarrow \quad x = c \quad \Rightarrow \quad |f(x) - f(c)| = 0$$

Theorem 19.3. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following conditions are equivalent.*

(a) *f is continuous at c.*
(b) *If (x_n) is a sequence in D converging to c, then $\lim_{n \to \infty} f(x_n) = f(c)$.*
(c) *For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.*

Furthermore, if c is an accumulation point of D, then the above are equivalent to

(d) *f has a limit at c and $lim_{x \to c} = f(c)$.*

Proof.

- *First suppose that c is an isolated point of D. In the remark that followed the definition of a continuous function we should that a function is always continuous at isolated points within its domain. So if c is an isolated point of D, (a) is always true. We now show that (c) is also always true in this case. For if c is an isolated point of D, then there exists a neighborhood $U = N(c, \delta)$ of c such that*

  $$U \cap D = \{c\}$$

  But then

  $$f(U \cap D) = f(\{c\}) = \{f(c)\}$$

which certainily lies within any neighborhood of $f(c)$. To see that (b) must also be true when c is an isolated point, let (x_n) be a sequence in D converging to c. Then there exists an N such that

$$\begin{aligned} n > N \quad &\Rightarrow \quad |x_n - c| < \delta \quad \Rightarrow \quad x_n \in U \cap D \\ &\Rightarrow \quad x = c \\ &\Rightarrow \quad f(x_n) - f(c) = 0 \\ &\Rightarrow \quad |f(x_n) - f(c)| < \varepsilon \text{ for all } n \end{aligned}$$

for any $\varepsilon > 0$. So then $\lim_{n\to\infty} f(x_n) = f(c)$.

- *Now assume c is an accumulation point of D. Then (a) $\Longleftrightarrow$ (d) is the definition of continuity at c, (d) $\Longleftrightarrow$ (c) is Theorem 20.2, and (d) $\Longleftrightarrow$ (b) is essentially Theorem 20.8.*

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THEOREM 19.4. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in D such that (x_n) converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

THEOREM 19.5. *Let f and g be functions from D to $\mathbb{R}$ and let $c \in D$. Suppose that f and g are continuous at c. Then*

1. *$(f + g)$ is continuous at c.*
2. *(fg) is continuous at c.*
3. *(f/g) is continous at c provided that $g(c) \neq 0$.*

THEOREM 19.6. *Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continous at a point $c \in D$ and g is continous at the point $f(c)$, then the composed function $g \circ f : D \to \mathbb{R}$ is continuous at c.*

Proof. (Homework problem).