LECTURE 18

Limits of Functions

DEFINITION 18.1. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a* ***limit of** f **at** c *if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$x \in D \quad \text{and} \quad 0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

NOTATION 18.2. We write

$$\lim_{x \to c} f(x) = L$$

to indicate that L is the limit of f at c.

REMARK 18.3. The definition effectively says that if $\lim_{x\to c} f(x)$=L we can make the values of f as close as we like to L by stipulating that x is sufficiently close to c. Note, however, that the value of f precisely at the point c is irrelevant. It is important to understand that the limit of a function is condition on the behavior of a function in a deleted neighborhood of a point, rather than a condition on its value at a particular point.

EXAMPLE 18.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

I claim $\lim_{x\to 1} f(x) = 1$ despite the fact that $f(1) = 0$. To see this, let ε be an arbitrary positive number. We have

$$\begin{aligned} 0 &< |x-1| < \delta \quad \Rightarrow x \in (1-\delta, 1+\delta) \text{ and } x \neq 1 \\ &\Rightarrow \quad (1-\delta)^2 < f(x) < (1+\delta)^2 \\ &\Rightarrow \quad \delta^2 - 2\delta < f(x) - 1 < \delta^2 + 2\delta \\ &\Rightarrow \quad |f(x) - 1| < \delta^2 + 2\delta \end{aligned}$$

We now choose δ so that

$$2\delta + \delta^2 \leq \varepsilon$$

e.g., solve $\delta^2 + 2\delta - \varepsilon = 0$ to get

$$\delta = \frac{-2 \pm \sqrt{4 + 4\varepsilon}}{2} = \sqrt{1+\varepsilon} - 1$$

Then we'll have

$$0 < |x-1| < \delta \quad \Rightarrow \quad |f(x) - 1| < \delta^2 + 2\delta = \varepsilon$$

and so

$$\lim_{x \to 1} f(x) = 1$$

∎

THEOREM 18.5. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ if and only if for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.*

THEOREM 18.6. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x\to c} f(x) = L$ if and only if for every sequence (s_n) in $D \backslash \{c\}$ that converges to c we have $\lim(f(s_n)) = L$.*

Proof.

$\Rightarrow$ Suppose that $\lim_{x\to c} f(x) = L$ and let (s_n) be sequence in $D\backslash\{c\}$ such that $\lim s_n = c$. By the definition of the limit of a function, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|f(x) - L| < \varepsilon$. Also since $\lim_{n\to\infty} s_n = c$, there exists an N such that $|s_n - c| < \delta$. Since each $s_n \neq c$ we have

$$n > N \quad \Rightarrow \quad 0 < |s_n - c| < \delta$$

and since each each $s_n \in D$, $f(s_n)$ is defined for each n, hence

$$n > N \quad \Rightarrow \quad 0 < |s_n - c| < \delta \quad \Rightarrow \quad |f(s_n) - L| < \varepsilon$$

so

$$\lim_{n\to\infty} f(s_n) = L$$

$\Leftarrow$ We want to show that if $\lim f(s_n) = L$ for every sequence (s_n) in $D\backslash\{c\}$ that converges to c, then L is the limit of f at c. We shall prove instead the contrapositive of this proposition:

- If L is not the limit of f at c, then there is a sequence $(s_n) \in D\backslash\{c\}$ such that $\lim (s_n) = L$.

Since $L \neq \lim_{x\to c} f(x)$, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there must exists an $x \in D$ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| \geq \varepsilon$$

Setting, successively $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ we obtain corresponding choices of x, $x_1, x_2, \ldots, x_n, \ldots \in D$ such that

$$0 < |x_n - c| < \frac{1}{n}$$

for all $n \in \mathbb{N}$ and

$$|f(x_n) - L| \geq \varepsilon$$

for all $n \in \mathbb{N}$. Setting $s_n = x_n$ we obtain a sequence of points of $D\backslash\{c\}$ that converges to c but for which

$$\lim |f(s_n)| \neq L$$

And so the contrapositive proposition is proved. ∎

Corollary 18.7. *If $f : D \to \mathbb{R}$ and c is an accumulation point of D, then f can have only one limit at c.*

Theorem 18.8. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following statements are equivalent:*

1. *f does not have a limit at c.*
2. *There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c but $(f(s_n))$ is not convergent in $\mathbb{R}$.*

Proof. (Homework)

Definition 18.9. *Let f and g be functions from D to $\mathbb{R}$. We define the* ***sum*** *$f + g$ to be the function from D to $\mathbb{R}$ defined by*

$$(f + g)(x) = f(x) + g(x)$$

We define the ***product*** *fg to be the function from D to $\mathbb{R}$ defined by*

$$(fg)(x) = f(x)g(x)$$

If $k \in \mathbb{R}$, we define the ***multiple*** *kf to be the function from D to $\mathbb{R}$ defined by*

$$(kf)(x) = kf(x)$$

If $g(x) \neq 0$ for all $x \in D$, we define the ***quotient*** *f/g to be the function from D to $\mathbb{R}$ defined by*

$$(f/g)(x) = f(x)/g(x)$$

THEOREM 18.10. *Let f and g be functions from D to $\mathbb{R}$ and let c be an accumulation point of D. Let $k \in \mathbb{R}$ and suppose*

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

then

$$\begin{aligned} \lim_{x \to c} (f + g) &= L + M \\ \lim_{x \to c} (fg) &= LM \\ \lim_{x \to c} (kf) &= kL \end{aligned}$$

Furthermore, if $g(x) \neq 0$ for all $x \in D$ and $M \neq 0$, then

$$\lim_{x \to c} (f/g) = L/M$$

1. One Sided Limits

It happens often that the domain D of a function is a open interval and one is interested in the behavior of the function as x approaches the boundary point of the interval from one side only. We sometimes indicate this by writing

$$\lim_{x \to c^+} f(x)$$

to indicate a limit of a function where the domain of the function has c as a left-most boundary point (e.g. $D = (c, +\infty)$). Similarly,

$$\lim_{x \to c^-} f(x)$$

indicates a limit where the domain D of f is bounded on the right by c (e.g. $D = (-\infty, c)$). You can thing of $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ as, respectively, the limits of f as x approaches c from the, respectively, positive side and negative side.