The Completeness Axiom, Cont’d

Axiom 12.1 (The Completeness Axiom). Every non-empty subset $S$ of $\mathbb{R}$ that is bounded from above has a least upper bound $\sup(S) \in \mathbb{R}$.

Theorem 12.1. Every non-empty subset $S$ of $\mathbb{R}$ that is bounded from below has a greatest lower bound $\inf S$.

Proof. Let $T$ be the set $\{ -s \mid s \in S \}$. Since $S$ is bounded from below there is an $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$. This implies $-s \leq -m$ for all $s \in S$ and so $t \leq -m$ for all $t \in T$. So $T$ is bounded from above, hence by the Completeness Axiom, $\sup T$ exists. Let $u = \sup T$. We shall show that $-u = \inf S$.

More precisely, we shall show that

$$-u \leq s, \quad \forall s \in S \tag{12.1}$$

and that

$$t \leq s, \quad \forall s \in S \Rightarrow t \leq -u \tag{12.2}$$

Now by definition, since $u$ is the least upper bound of $T$,

$$-s \leq u, \quad \forall s \in S \tag{12.3}$$

and

$$-s \leq q, \quad \forall s \in S \Rightarrow u \leq q \tag{12.4}$$

Now from Theorem 3.2 (i) we know (12.3) is equivalent to (12.1). Setting $q = -t$, (12.4) reads

$$-s \leq -t, \quad \forall s \in S \Rightarrow u \leq -t$$

or, using Theorem 3.4 (i) again,

$$t \leq s, \quad \forall s \in S \Rightarrow t \leq -u$$

which is precisely (12.2).

□

Theorem 12.2 (The Archimedean Property of $\mathbb{R}$). The set $\mathbb{N}$ of natural numbers is unbounded from above in $\mathbb{R}$.

Proof. (Proof by Contradiction). Suppose $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Then by the Completeness Axiom, $\mathbb{N}$ has a least upper bound $m \in \mathbb{R}$. This implies that $m - 1$ is not an upper bound for $\mathbb{N}$ (since there can be no upper bound smaller than $m$), hence the must be an element $n \in \mathbb{N}$ such that

$$m - 1 < n$$

But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$ and so adding 1 to both sides of the above inequality yields

$$m - 1 + 1 < n + 1 \in \mathbb{N}$$

so $m$ cannot be an upper bound for $\mathbb{N}$ (let alone the least upper bound). We conclude that $m = \sup(\mathbb{N})$ does not exist.

Theorem 12.3. The following statements are equivalent to the Archimedean Property of $\mathbb{R}$.

(1) For each $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$. 

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(2) If \( x > 0 \) and for each \( y \in \mathbb{R} \), there exists an \( n \in \mathbb{N} \) such that \( nx > y \).

(3) For each \( x > 0 \), there exists \( n \in \mathbb{N} \) such that \( 0 < \frac{1}{n} < x \).

**Proof.**

(Archimedian Property \( \Rightarrow \) 1). Suppose (1) is false. Then there exists a \( z \in \mathbb{R} \), such that no \( n \in \mathbb{N} \) is such that \( n > z \); i.e., \( n \leq z \) for all \( n \in \mathbb{N} \), i.e. \( \mathbb{N} \) has an upper bound in \( \mathbb{R} \). Hence the Archimedian Property is false. Thus, the contrapositive of (Archimedian Property \( \Rightarrow \) 1) is proven.

(1 \( \Rightarrow \) 2). Let \( z = y/x \). Then, if (1) is true, there exists \( n \in \mathbb{N} \) such that \( n > \frac{y}{x} \), or (using that fact that \( x > 0 \)) that \( nx > y \).

(2 \( \Rightarrow \) 3). Suppose (2) is true. Then setting \( y = 1 \) we know there exists \( n \) such that \( nx > 1 \). Multiplying both sides of this last inequality by \( 1/n \) we have \( \frac{1}{n} < x \). Also, \( 0 < \frac{1}{n} \) since if it were false, then \( \frac{1}{n} \leq 0 \). And this last inequality when multiplied by the positive number \( n^2 \) would yield \( n \leq 0 \) which would mean that \( n \) was not a positive integer.

(3 \( \Rightarrow \) Archimedian Property). Suppose that \( \mathbb{N} \) is bounded above by some real number \( m \); i.e., \( n < m \) for all \( n \in \mathbb{N} \). Then

\[
\frac{1}{m} < \frac{1}{n} \quad \forall n \in \mathbb{N}
\]

which contradicts (3) (because there'd be no \( 1/n \) between 0 and \( \frac{1}{m} \)). Thus the contrapositive of (3 \( \Rightarrow \) Archimedian Property) is proven.

**Theorem 12.4. (The Denseness of \( \mathbb{Q} \))** If \( a, b \in \mathbb{R} \) and \( a < b \), then there is a rational number \( r \) such that \( a < r < b \).

**Proof.** It suffices to show that there exist integers \( m \) and \( n > 0 \) such that

\[
a < \frac{m}{n} < b
\]

Since \( 0 < b - a \), the Statement (2) of the preceding theorem tells us that there exists an \( n \in \mathbb{N} \) such that

\[
1 < n(b - a)
\]

or

\[
a(n + 1) < bn
\]

At this point it seems obvious that there is an integer lying between \( an \) and \( bn \). Rather than make a plausibility argument, we shall provide an explicit construction of such an integer.

By the Archimedian Property again, there also exists positive integers \( k', k'' \) such that

\[
|an| < k' \quad \text{and} \quad |bn| < k''
\]

Set

\[
k = \max\{k', k''\}
\]

Then

\[
-k < an < bn < k
\]

The set

\[
\{j \in \mathbb{Z} \mid -k < j < k \text{ and } an < j\}
\]

is finite and nonempty. Set

\[
m = \min\{j \in \mathbb{Z} \mid -k < j < k \text{ and } an < j\}
\]

so that

\[
an < m \quad \text{but} \quad m - 1 \leq an.
\]
Then we have

\[ m = (m - 1) + 1 \leq an + 1 < bn \]

Now we have found an \( m \in \mathbb{Z} \) such that

\[ an < m < bn \]

\( \square \)