Section 7.1

7.1.1. Evaluate \( \int_{\sigma} f \, ds \) where \( f(x, y, z) = x + y + z \) and \( \sigma : t \mapsto (\sin(t), \cos(t), t), \ t \in [0, 2\pi] \).

- We have
  \[
  \frac{d\sigma}{dt} = (\cos(t), -\sin(t), 1)
  \]
  Thus,
  \[
  \left\| \frac{d\sigma}{dt} \right\| = \sqrt{\cos^2(t) + \sin^2(t) + 1} = \sqrt{2}
  \]
  And so
  \[
  \int_{\sigma} f \, ds = \int_{0}^{2\pi} f(\sigma(t)) \left\| \frac{d\sigma}{dt}(t) \right\| \, dt
  = \int_{0}^{2\pi} (\cos(t) + \sin(t) + t) \sqrt{2} \, dt
  = \sqrt{2} \left( \sin(t) - \cos(t) + \frac{1}{2}t^2 \right) \bigg|_{0}^{2\pi}
  = 2\sqrt{2}\pi^2
  \]

7.1.2. Evaluate the path integral
  \[
  \int_{C} f \, ds
  \]
  where \( f(x, y, z) = yz \) and \( C \) is the curve parameterized by \( \sigma : t \mapsto (t, 3t, 2t), \ t \in [1, 3] \).

- Now
  \[
  \frac{d\sigma}{dt} = (1, 3, 2)
  \]
  And so
  \[
  \left\| \frac{d\sigma}{dt} \right\| = \sqrt{1 + 9 + 4} = \sqrt{14}
  \]
  We have
  \[
  \int_{C} f \, ds
  = \int_{1}^{3} (3t)(2t)\sqrt{14} \, dt
  = \int_{1}^{3} 6\sqrt{14}t^2 \, dt
  = 2\sqrt{14} \left[ t^3 \right]_{1}^{3}
  = 52\sqrt{14}
  \]
Section 7.2

7.2.1. Let $F(x, y, z) = xi + yj + zk$. Evaluate the line integral of $F$ along the path $\sigma(t) = (t, t, t)$, $0 \leq t \leq 1$.

- We have

$$\int_{\sigma} F \cdot ds = \int_{0}^{1} F(t, t, t) \cdot \frac{d\sigma}{dt} dt$$

$$= \int_{0}^{1} (t, t, t) \cdot (1, 1, 1) dt$$

$$= \int_{0}^{1} 3t dt$$

$$= \frac{3}{2} \blacksquare$$

7.2.2. Consider the force $F(x, y, z) = xi + yj + zk$. Compute the work done in moving along the parabola $y = x^2$, $z = 0$, from $x = -1$ to $x = 2$.

- We can parameterize the parabola via the path

$$\sigma(t) = (t, t^2, 0), \quad t \in [-1, 2]$$

We then have

$$\int_{\sigma} F \cdot ds = \int_{-1}^{2} F(t, t^2, 0) \cdot \frac{d\sigma}{dt}(t)dt$$

$$= \int_{-1}^{2} (t, t^2, 0) \cdot (1, 2t, 0) dt$$

$$= \int_{-1}^{2} (t + 2t^3) dt$$

$$= \left( \frac{1}{2}t^2 + \frac{1}{2}t^4 \right) \bigg|_{-1}^{2}$$

$$= 2 + 8 - \frac{1}{2} - \frac{1}{2}$$

$$= 9 \blacksquare$$
Section 7.3

7.3.1. Find the equation of the tangent plane to the parameterized surface $\Phi(u, v) = (2u, u^2 + v, v^2)$ at the point $(0,1,1)$.

- The values of $u$ and $v$ corresponding to the point $(0,1,1)$ are $u = 0$ and $v = 1$. The normal vector to the tangent plane at $(0,1,1)$ is thus given by

$$
\mathbf{n}(0,1) = \left. \frac{\partial \Phi}{\partial u} \right|_{(0,1)} \times \left. \frac{\partial \Phi}{\partial v} \right|_{(0,1)} \\
= (2, 2u, 0) \times (0, 1, 2v) \bigg|_{(0,1)} \\
= (2, 0, 0) \times (0, 1, 2) \\
= (0 - 0, 0 - 4, 2 - 0) \\
= (0, -4, 2)
$$

The equation of the plane through the point $(0,1,1)$ normal to the direction $\mathbf{n} = (0, -4, 2)$ is

$$
0 = \mathbf{n} \cdot ((x, y, z) - (0, 1, 1)) = -4(y - 1) + 2(z - 1) = 2(-2y + 1 + z)
$$

or

$$
z = 2y - 1
$$

7.3.2. Find an expression for the unit vector normal to the parameterized surface

$$
\Phi(u, v) = (\cos(v) \sin(u), \sin(v) \sin(u), \cos(u)) \quad , \quad (u, v) \in [0, \pi] \times [0, 2\pi]
$$

Identify this surface.

- A normal vector to surface at the point $\Phi(u, v)$ is given by

$$
\mathbf{n}(u, v) = \left. \frac{\partial \Phi}{\partial u} \right|_{(u,v)} \times \left. \frac{\partial \Phi}{\partial v} \right|_{(u,v)} \\
= \left( \cos(v) \cos(u), \sin(v) \cos(u), -\sin(u) \right) \times \left( -\sin(v) \sin(u), \cos(v) \sin(u), 0 \right) \\
= \left( 0 + \cos(v) \sin^2(u), \sin(v) \sin^2(u) - 0, \cos(u) \sin(u) \cos^2(v) - \cos(u) \sin(u) \sin^2(v) \right) \\
= \left( \cos(v) \sin^2(u), \sin(v) \sin^2(u), \cos(u) \sin(u) \right) \\
= \sin(u) \left( \cos(v) \sin(u), \sin(v) \sin(u), \cos(u) \right)
$$

We have

$$
\|\mathbf{n}\|^2 = \sin^2(u) \left( \cos^2(v) \sin^2(u) + \sin^2(u) \sin^2(v) + \cos^2(u) \right) \\
= \sin^2(u) \left( \sin^2(u) + \cos^2(u) \right) \\
= \sin^2(u)
$$

Therefore, the unit normal vector at $\Phi(u,v)$ is

$$
\frac{\mathbf{n}(u, v)}{\|\mathbf{n}(u, v)\|} = (\cos(v) \sin(u), \sin(v) \sin(u), \cos(u))
$$

The surface $S$ is that of the unit sphere.
Section 7.4

7.4.1. Find the surface area of the unit sphere \( S \) represented parametrically by

\[
\Phi(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)), \quad (\theta, \phi) \in [0, 2\pi] \times [0, \pi].
\]

- In the problem 7.3.5, we calculated the normal vector \( \mathbf{n} \) to the surface prescribed by \( \Phi \). It remains to calculate the integral of \( \| \mathbf{n} \| \) over the domain of \( \Phi \)

\[
A(S) = \int_D \| \mathbf{n} \| dS
\]

\[
= \int_{0}^{\pi} \int_{0}^{2\pi} \sin(\phi) d\theta d\phi
\]

\[
= \int_{0}^{\pi} 2\pi \sin(\phi)
\]

\[
= 2\pi \int_{0}^{\pi} \cos(\phi)
\]

\[
= 2\pi (-(-1) + 1)
\]

\[
= 4\pi
\]

\[
\square
\]

7.4.2. Let \( \Phi(u, v) = (u - v, u + v, uv) \) and let \( D \) be the unit disk in the \( uv \) plane. Find the area of \( \Phi(D) \).

- We have

\[
\mathbf{n}(u, v) = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}
\]

\[
= (1, 1, v) \times (-1, 1, u)
\]

\[
= (u - v, -v - u, 2)
\]

Thus,

\[
\| \mathbf{n} \| = \sqrt{(u - v)^2 + (-v - u)^2 + 4}
\]

\[
= \sqrt{2(u^2 + v^2 + 2)}
\]

and

\[
A(S) = \int_D \| \mathbf{n} \| dA
\]

\[
= \int_D \sqrt{2(u^2 + v^2 + 2)} dA
\]

To carry out this integral over the unit disk, we make a change of variables to polar coordinates

\[
u = r \cos(\theta)
\]

\[
v = r \sin(\theta)
\]
Recall that the Jacobian of the this transformation is \( r \). Thus,

\[
A(S) = \int_D \sqrt{2 (u^2 + v^2 + 2)} dA \\
= \int_0^1 \int_0^{2\pi} \sqrt{2 (r^2 + 2)} r \, d\theta \, dr \\
= \int_0^1 2\pi \sqrt{2 (r^2 + 2)} r \, dr \\
= 2\pi \int_1^2 \sqrt{\frac{r^2}{4}} \, dr \\
= \left[ \frac{2}{3} \sqrt{r^2} \right]_1^2 \\
= \frac{\pi}{3} \left( 6\sqrt{6} - 8 \right)
\]
Section 7.5

7.5.1. Evaluate \( \int_S z \, dS \) where \( S \) is the upper hemisphere of radius \( a \), that is, the set
\[
\left\{ (x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{a^2 + x^2 + y^2} \right\}
\]

- We parameterize the upper hemisphere \( S \) via the map
\[
\Phi(\theta, \phi) = (a \cos(\theta) \sin(\phi), a \sin(\theta) \sin(\phi), a \cos(\phi))
\]
with \( 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \frac{\pi}{2} \). We have already calculated the normal vector \( n \) associated with the analogous parameterization of the unit sphere. (See problem 7.3.5.) Reviewing that calculation, it becomes obvious that for a (hemi-) sphere of radius \( a \) we would have
\[
\|n\| = a^2 \sin(\phi)
\]
Thus,
\[
\int_S z \, dS = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} a \cos(\phi) \|n\| d\theta \, d\phi
\]
\[
= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} a^3 \cos(\phi) \sin(\phi) d\theta \, d\phi
\]
\[
= 2\pi \int_0^{\frac{\pi}{2}} a^3 \sin(\phi) \cos(\phi) \, d\phi
\]
\[
= 2\pi a^3 \int_0^1 u \, du
\]
\[
= \pi a^3
\]
Section 7.6

7.6.1. Let the temperature of a point in $\mathbb{R}^3$ be given by $3x^2 + 3z^2$. Compute the heat flux across the surface $x^2 + z^2 = 2$, $0 \leq y \leq 2$ if $k = 1$

- According to the text, the heat will flow with the vector field
  \[ \mathbf{F} = -k\nabla T \]
  and so the heat flux across the surface $S$ defined above, with $k = 1$, will be
  \[ \int_S \mathbf{F} \cdot \mathbf{dS} = -\int_S \nabla T \cdot \mathbf{dS} \]
  To calculate the integral on the right hand side we need to first find a suitable parameterization of the surface $S$. It is pretty obvious that $S$ is just the surface of a cylinder of radius $\sqrt{2}$ with axis of symmetry coinciding along the $y$ axis and with its top and bottom removed. We therefore use cylindrical coordinates to parameterize $S:
  \Phi(\theta, y) = \left(\sqrt{2}\cos(\theta), y, \sqrt{2}\sin(\theta)\right) \quad (\theta, y) \in [0, 2\pi] \times [0, 2]
  
  We then have
  \[ \mathbf{n}(\theta, y) = \left(\frac{\partial \Phi}{\partial \theta}|_{(\theta, y)} \times \frac{\partial \Phi}{\partial y}|_{(\theta, y)}\right) \]
  \[ = \left( -\sqrt{2}\sin(\theta), 0, \sqrt{2}\cos(\theta) \right) \times (0, 1, 0) \]
  \[ = \left( -\sqrt{2}\cos(\theta), 0, -\sqrt{2}\sin(\theta) \right) \]
  Hence,
  \[ \int_S \mathbf{F} \cdot \mathbf{dS} = \int_S -\nabla T \cdot \mathbf{dS} \]
  \[ = \int_S (-6x, 0, -6z) \cdot \mathbf{dS} \]
  \[ = \int_0^2 \int_0^{2\pi} \left( -6\sqrt{2}\cos(\theta), 0, -6\sqrt{2}\sin(\theta) \right) \cdot \left( -\sqrt{2}\cos(\theta), 0, -\sqrt{2}\sin(\theta) \right) d\theta dy \]
  \[ = \int_0^2 \int_0^{2\pi} \left( 12\cos^2(\theta) + 12\sin^2(\theta) \right) d\theta dy \]
  \[ = \int_0^2 \int_0^{2\pi} 12 \, d\theta \, dy \]
  \[ = \int_0^2 24\pi \, dy \]
  \[ = 48\pi \]

\[ \square \]