Section 6.1

6.1.1. Let $S^* = (0, 1] \times [0, 2\pi)$ and defined $T(r, \theta) = (r \cos(\theta), r \sin(\theta))$. Determine the image set $S$ and show that $T$ is one-to-one on $S^*$.

- To find the image of $S^*$ under $T$, we first calculate the image of the boundary of $S^*$.

Now the boundary of $S^*$ consists of the the following 4 curves:

$$
\begin{align*}
\sigma_1(t) &= (t, 0), \quad t \in [0, 1] \\
\sigma_2(t) &= (1, t), \quad t \in [0, 2\pi] \\
\sigma_3(t) &= (1 - t, 2\pi), \quad t \in [0, 1] \\
\sigma_4(t) &= (0, 2\pi - t), \quad t \in [0, 2\pi]
\end{align*}
$$

Note the curves $\sigma_3$ and $\sigma_4$, while part of the boundary of $S^*$ do not belong to $S^*$.

The images of these four curves under the map $T$ are given by

$$
\begin{align*}
\gamma_1(t) &= T(\sigma_1(t)) = (t, 0), \quad t \in [0, 1] \\
\gamma_2(t) &= T(\sigma_2(t)) = (\cos(t), \sin(t)), \quad t \in [0, 2\pi] \\
\gamma_3(t) &= T(\sigma_3(t)) = (1 - t, 0), \quad t \in [0, 1] \\
\gamma_4(t) &= T(\sigma_4(t)) = (0, 0), \quad t \in [0, 2\pi]
\end{align*}
$$

Thus, the image of $\gamma_1$ is portion of the x-axis between 0 and 1, the image of $\gamma_1$ is the unit circle, the image of $\gamma_3$ is the portion of the x-axis between 0 and 1, and the image of $\gamma_4$ is the origin.

It would appear that the portion of the x-axis between 0 and 1 is counted twice - however recall that the curve $\sigma_3$ does not lie in the domain of $T$. Nor does the curve $\sigma_4$. Therefore, the image curves $\gamma_3$ and $\gamma_4$ are not to be considered as being part of $S$. We conclude that the image of $S^*$ by $T$ is the unit disk minus the origin.

To show that the map $T$ is one-to-one, we must show (i) that $T$ is surjective; i.e., every point of the unit disc is the image of the form $(x, y) = T(r, \theta)$ for some $(r, \theta) \in S^*$.

This is already evident from the definition of $S$: $S := \{ (x, y) \in \mathbb{R}^2 | (x, y) = T(r, \theta), \quad \text{for some } (r, \theta) \in S^* \}$.

(ii) that $T$ is injective; i.e., if $T(r, \theta) = T(r', \theta')$ then $(r, \theta) = (r', \theta')$.

Well, suppose

$$(r \cos(\theta), r \sin(\theta)) = (r' \cos(\theta'), r' \sin(\theta'))$$

This can happen if and only if

$$r = r' = 0$$

or

$$r = r' \quad \text{and} \quad \theta = \theta' + 2\pi n \quad , \quad n \in \mathbb{Z}$$

But $r = r' = 0$ is excluded from $S^*$, and there are no two $(r, \theta), (r', \theta')$ in $S^*$ for which $\theta = \theta' + 2\pi n$.

Hence, the transformation $T$ is one-to-one.

6.1.2. Let $D^* = [0, 1] \times [0, 1]$ and define $T$ on $D^*$ by $T(u, v) = (-u^2 + 4u, v)$. Find $D$. Is $T$ one-to-one?

- The region $D^*$ is the region in the $uv$-plane bounded by the lines

$$
\begin{align*}
\sigma_1(t) &= (t, 0), \quad t \in [0, 1] \\
\sigma_2(t) &= (1, t), \quad t \in [0, 1] \\
\sigma_3(t) &= (t, 1), \quad t \in [0, 1] \\
\sigma_4(t) &= (0, t), \quad t \in [0, 1]
\end{align*}
$$

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The region $D = T(D)$ should therefore be the region in the $xy$-plane bounded by the curves

$$
\gamma_1(t) = T(\sigma_1(t)) = (-t^2 + 4t, 0), \quad t \in [0, 1]
$$

$$
\gamma_2(t) = T(\sigma_2(t)) = (3, t), \quad t \in [0, 1]
$$

$$
\gamma_3(t) = T(\sigma_3(t)) = (-t^2 + 4t, 1), \quad t \in [0, 1]
$$

$$
\gamma_4(t) = T(\sigma_4(t)) = (0, t), \quad t \in [0, 1]
$$

The curve $\gamma_1$ is the line segment along the $x$-axis between $(0, 0)$ and $(3, 0)$, $\gamma_2$ corresponds to the vertical line segment between the points $(3, 0)$ and $(3, 1)$, $\gamma_3$ corresponds to the horizontal line segment between the points $(3, 1)$ and $(0, 1)$, and $\gamma_4$ corresponds to the vertical line segment between $(0, 0)$ and $(0, 1)$.

Thus, $S = [0, 3] \times [0, 1]$.

By definition $T : S^* \rightarrow S$ is surjective. We check to see that $T$ is injective. If

$$
(-u^2 + 4u, v) = \left(-u'^2 + 4u', v'\right)
$$

then we must have

$$
u' = \frac{4+\sqrt{16-4v^2-4u}}{2} = 2 \pm (u - 2) = \begin{cases} u & \text{if } u \in [0, 1], v' = 4 - u \neq [0, 1]. \end{cases}
$$

But if $u \in [0, 1], v' = 4 - u \neq [0, 1]$. So it is not possible for two distinct points in $S^*$ to land on the same point in $S$ under the map $T$. Thus, $T$ is both surjective and injective; hence $T$ is one-to-one.

6.1.3. Let $D^* = [0, 1] \times [0, 1]$ and define $T$ on $D^*$ by $T(u, v) = (uv, u)$. Find $D$. Is $T$ one-to-one? If not, can we eliminate some subset of $D^*$ so that on the remainder $T$ is one-to-one?

- The region $D^*$ is the region in the $wv$-plane bounded by the lines

$$
\sigma_1(t) = (t, 0)
$$

$$
\sigma_2(t) = (1, t)
$$

$$
\sigma_3(t) = (t, 1)
$$

$$
\sigma_4(t) = (0, t)
$$

The region $D = T(D)$ should therefore be the region in the $xy$-plane bounded by the curves

$$
\gamma_1(t) = T(\sigma_1(t)) = (0, t)
$$

$$
\gamma_2(t) = T(\sigma_2(t)) = (t, 1)
$$

$$
\gamma_3(t) = T(\sigma_3(t)) = (t, 1)
$$

$$
\gamma_4(t) = T(\sigma_4(t)) = (0, 0)
$$

Thus, the image of the curve $\sigma_1$ coincides with the $y$-axis; the image of the curve $\sigma_2$ is just the horizontal line $y = 1$; the image of the curve $\sigma_3$ coincides with the diagonal line $y = x$; and image of the curve $\sigma_4$ is just the point $(0, 0)$. Thus, $D$ coincides with the interior of the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

This map cannot be one-to-one since every point along the curve $\sigma_4$ is mapped to the point $(0, 0)$. However, if we remove this curve from the domain of $T$, then the map becomes one-to-one.

6.1.4. Let $T(x) = Ax$ where $A$ is a $2 \times 2$ matrix. Show that $T$ is one-to-one if and only if the determinant of $A$ is non-zero.

- If $T$ is one-to-one then it must have an inverse. However, a matrix $A$ has an inverse if and only if its determinant is non-zero. Thus, $T$ is one-to-one if and only if $det A \neq 0$.

6.1.5. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear; i.e., $T(x) = Ax$, where $A$ is a $2 \times 2$ matrix. Show that if $det A \neq 0$, then $T$ takes parallelograms to parallelograms. (Hint: any parallelogram in $\mathbb{R}^2$ can be described as a set $\{r = p + \lambda v + \mu w \mid \lambda, \mu \in [0, 1]\}$ where $p, v, w$ are suitable vectors in $\mathbb{R}^2$ with $v$ not a scalar multiple of $w$. )
• Suppose

\[ P = \{ r \in \mathbb{R}^2 \mid r = p + \lambda v + \mu w \mid \lambda, \mu \in [0,1] \} \]

is a parallelogram. Then \( T(P) \) is

\[
\begin{align*}
T(P) &= \{ r' \in \mathbb{R}^2 \mid r' = A(p + \lambda v + \mu w) \mid \lambda, \mu \in [0,1] \} \\
&= \{ r' \in \mathbb{R}^2 \mid r' = Ap + A(\lambda v) + A(\mu w) \mid \lambda, \mu \in [0,1] \} \\
&= \{ r' \in \mathbb{R}^2 \mid r' = (Ap) + \lambda(Av) + \mu(Aw) \mid \lambda, \mu \in [0,1] \} \\
&= \{ r' \in \mathbb{R}^2 \mid r' = p' + \lambda v' + \mu w' \mid \lambda, \mu \in [0,1] \}
\end{align*}
\]

where \( p' = Ap, v' = Av, \) and \( w' = Aw \). If we can demonstrate that \( v' \neq tw' \), for any \( t \in \mathbb{R} \), then we may conclude that \( T(P) \) is a parallelogram.

We argue as follows. Suppose \( v' = tw' \). Then

\[
0 = v' - tw = Av - tAw = A(v - tw)
\]

Since \( v \) and \( w \) are not scalar multiples of one another, \( v - tw \) must be a non-zero vector. But if the matrix \( A \) maps a non-zero vector to zero, it must be singular; hence, \( \det A = 0 \). But hypothesis, \( \det A \neq 0 \). Therefore, \( v' \neq tw' \). Hence, \( T(P) \) is a parallelogram. \( \square \)
Section 6.2

6.2.1. Let $D$ be the unit circle. Evaluate
\[ \int_D \exp(x^2 + y^2) \, dx \, dy \]
by making a change of variables to polar coordinates.

- The coordinate transformation
  \[ T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta)) \]
maps the rectangle $R = \{ 0 \leq r \leq 1, \ 0 \leq \theta < 2\pi \}$ onto the unit circle. The Jacobian of this transformation is

\[ J(T) = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| = \left| (\cos(\theta)) (r \cos(\theta)) - (\sin(\theta)) (-r \sin(\theta)) \right| = \left| r (\cos^2(\theta) + \sin^2(\theta)) \right| = r \]

Thus, by the change of variables formula
\[
\int_D \exp(x^2 + y^2) \, dx \, dy = \int_R \exp(r^2) \, J(T) \, dr \, d\theta
\]
\[
= \int_0^1 \int_0^{2\pi} e^{r^2} r \, d\theta \, dr
= 2\pi \int_0^1 \int_0^1 e^u \, du
= 2\pi \left( \frac{1}{2} e - 1 \right)
= \pi (e - 1)
\]

6.2.2. Let $D$ be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate
\[ \int_D (x + y) \, dx \, dy \]
by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

- Let $T$ be the coordinate transformation defined by
  \[ T : (u, v) \mapsto (u + v, u - v) \]
The Jacobian of this transformation is
\[
J(T) = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| (1)(-1) - (1)(1) \right| = 2
\]
To find the pre-image $D^*$ of the region $D$ by $T$, we first calculate the inverse of $T$. Solving
\[
x = u + v \\
y = u - v
\]
for $u$ and $v$ yields
\[
  u = \frac{1}{2} (x + y) \\
  v = \frac{1}{2} (x - y)
\]
Thus, the pre-images of the three boundary curves of $D$
\[
  \sigma_1(t) : t \mapsto (t,0), \quad t \in [0,1] \\
  \sigma_2(t) : t \mapsto (1,t), \quad t \in [0,1] \\
  \sigma_3(t) : t \mapsto (t,t), \quad t \in [0,1]
\]
will be
\[
  \gamma_1(t) = T^{-1}(\sigma_1) = \left( \frac{1}{2}, \frac{1}{2}t \right), \quad t \in [0,1] \\
  \gamma_2(t) = T^{-1}(\sigma_2) = \left( \frac{1}{2}(1+t), \frac{1}{2}(1-t) \right), \quad t \in [0,1] \\
  \gamma_3(t) = T^{-1}(\sigma_3) = (t,0), \quad t \in [0,1]
\]
The area in the $uv$-plane bounded by these three lines will be the triangle with vertices $(0,0)$, \(\left( \frac{1}{2}, \frac{1}{2} \right)\), and $(1,0)$.
This region can be regarded as a region of type II.
\[
  D^* = \left\{ 0 \leq v \leq \frac{1}{2}, \quad v \leq u \leq 1-v \right\}
\]
Thus,
\[
  \int_D (x+y) dx \; dy = \int_{D^*} 2u J(T) du \; dv \\
  = \int_0^{\frac{1}{2}} \int_{v}^{1-v} 4u \; du \; dv \\
  = \int_0^{\frac{1}{2}} \left[ \frac{2}{3} \left( 1 - 2v \right)^3 \right] \; dv \\
  = \frac{1}{2} \left( 1 - \frac{1}{2} \right) - 0 - 0 \\
  = \frac{1}{2}
\]
To check our result we shall integrate over $D$ directly. Now
\[
  D = \{ 0 \leq x \leq 1, \quad 0 \leq y \leq x \}
\]
so
\[
  \int_D (x+y) dx \; dy = \int_0^1 \int_0^x (x+y) \; dy \; dx \\
  = \int_0^1 \left[ xy + \frac{1}{2}y^2 \right]_0^x \; dx \\
  = \int_0^1 \left( \frac{3}{2}x^2 \right) \; dx \\
  = \frac{1}{2}
\]
6.2.3. Let \( T(u,v) = (x(u,v), y(u,v)) \) be the mapping defined by \( T(u,v) = (4u, 2u + 3v) \). Let \( D^* \) be the region in \( u - v \) plane corresponding to the rectangle \([0,1] \times [1,2]\). Find \( D = T(D^*) \) and evaluate

(a) \( \int_D xy \, dA \)

(b) \( \int_D (x - y) \, dA \)

- To find the image \( D \) of the region \( D^* \) in the \( xy \)-plane, we note the map \( T(u,v) \) is linear in \( u \) and \( v \). In Problem 6.2.10 we showed the image of a parallelogram in \( \mathbb{R}^2 \) (in particular, the image of a rectangle) by a linear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is always a parallelogram. To prescribe the image \( D \) of \( D^* \) by the map \( T \) is therefore sufficient to present its four vertices. But these vertices will just be the images of the corners of \( D^* \) by \( T \); thus, the region \( D \) will be the parallelogram in the \( xy \)-plane with vertices

\[
\begin{align*}
  \mathbf{v}_1 &= T(0,1) = (0,3) \\
  \mathbf{v}_2 &= T(1,1) = (4,5) \\
  \mathbf{v}_3 &= T(1,2) = (4,8) \\
  \mathbf{v}_4 &= T(0,2) = (0,6)
\end{align*}
\]

Let us now compute the Jacobian of the transformation.

\[
J(T) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \begin{vmatrix}
4 & 3 \\
2 & 0
\end{vmatrix} = 12
\]

Thus,

\[
\int_D xy \, dx \, dy = \int_{D^*} (4u)(2u + 3v)J(T) \, du \, dv
\]

\[
\begin{align*}
&= 24 \int_0^1 \int_1^2 (4u^2 + 6uv) \, dv \, du \\
&= 24 \int_0^1 \left[ 4u^2v + 3uv^2 \right]_1^2 \, du \\
&= 24 \int_0^1 (8u^2 + 12u - 4u^2 - 3u) \, du \\
&= 24 \left[ \frac{4}{3} u^3 + 9 \right]_0^1 \\
&= 32 + 108 \\
&= 140
\end{align*}
\]
\[
\int_D (x - y) \, dx \, dy = \int_{D^*} (4u - 2u - 3v) J(T) \, du \, dv \\
= 12 \int_0^1 \int_0^1 (2u - 3v) \, dv \, du \\
= 12 \int_0^1 \left( 2uv - \frac{3}{2} v^2 \right) \bigg|_0^1 \, du \\
= 12 \int_0^1 \left( 4u - 6 + 2u + \frac{3}{2} \right) \, du \\
= 12 \int_0^1 \left( u^2 - \frac{9}{2} \right) \, du \\
= 12 - 54 \\
= -42
\]

\[\square\]

6.2.4. Define \( T(u, v) = (u^2 - v^2, 2uv) \). Let \( D^* \) be the set of \((u, v)\) with \( u^2 + v^2 \leq 1 \), \( u \geq 0 \), \( v \geq 0 \). Find \( T(D^*) = D \). Evaluate

\[
\int_D dA
\]

- To find the image of \( D^* \), we calculate the images of the boundary curves of \( D^* \). The following three curves form the boundary of \( D^* \):
  \[
  \sigma_1(t) = (t, 0) \quad t \in [0, 1] \\
  \sigma_2(t) = (\cos(t), \sin(t)) \quad t \in \left[0, \frac{\pi}{2}\right] \\
  \sigma_3(t) = (0, t) \quad t \in [0, 1]
  \]

The images of these curves under the map \( T \) are given by

\[
\gamma_1(t) = T(\sigma_1(t)) = (t^2, 0) \quad t \in [0, 1] \\
\gamma_2(t) = T(\sigma_2(t)) = (\cos^2(t) - \sin^2(t), 2\cos(t)\sin(t)) \quad t \in \left[0, \frac{\pi}{2}\right] \\
\gamma_3(t) = T(\sigma_3(t)) = (-t^2, 0) \quad t \in [0, 1]
\]

These curves bound the region \( D \) pictured below:

The Jacobian of the transformation \( T \) is

\[
J(t) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}
\end{vmatrix}
= \left| (2u)(2u) - (-2v)(2v) \right|
= 4(u^2 + v^2)
\]

Thus,

\[
\int_D dy \, dx = \int_{D^*} J(T) \, du \, dv = \int_{D^*} 4(u^2 + v^2) \, du \, dv
\]

Since \( D^* \) is the unit disc, this last integral will be evaluated most easily if we make another change of variables to polar coordinates:

\[
u = r \cos(\theta) \\
v = r \sin(\theta)
\]
The Jacobian of this transformation is $r$. Thus,

$$
\int_{D^*} 4(u^2 + v^2) \, du \, dv = \int_0^1 \int_0^{2\pi} (4r^2) \, r \, d\theta \, dr
$$

$$
= 8\pi \int_0^1 r^3 \, dr
$$

$$
= 2\pi
$$

\[\square\]

6.2.5. Let $T(u,v)$ be as in Exercise 6.2.4. By making this change of variables, evaluate

$$
\int_D \frac{dA}{\sqrt{x^2 + y^2}}
$$

- The coordinate transformation in 6.2.4 is given by

$$
x = u^2 - v^2
$$

$$
y = 2uv
$$

and so the associated Jacobian is

$$
J(T) = \left| \det \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} \right|
$$

$$
= \left| \det \begin{pmatrix}
2u & -2v \\
2v & 2u
\end{pmatrix} \right|
$$

$$
= 2u^2 + 2v^2
$$

$$
\int_D \frac{dA}{\sqrt{x^2 + y^2}} = \int_{D^*} \frac{1}{\sqrt{(u^2 - v^2) + (2uv)^2}} J(T)
$$

$$
= \int_{-1}^1 \int_0^{\sqrt{1-u^2}} \frac{2u^2 + 2v^2}{\sqrt{u^4 + 2u^2v^2 + v^4}} \, dv \, du
$$

$$
= \int_{-1}^1 \int_0^{\sqrt{1-u^2}} \frac{2u^2 + 2v^2}{\sqrt{(u^2 + v^2)^2}} \, dv \, du
$$

$$
= \int_{-1}^1 2 \sqrt{1-u^2} \, du
$$

$$
= \left. \left( x \sqrt{(1-x^2) + \arcsin(x)} \right) \right|_{-1}^1
$$

$$
= 0 + \frac{\pi}{2} - \left( 0 + \left( -\frac{\pi}{2} \right) \right)
$$

$$
= \pi
$$

\[\square\]

6.2.6. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \leq 4$, $-2 \leq z \leq 3$.

- The cylinder $D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4, \, -2 \leq z \leq 3\}$ is the image of the rectangle $D^* = \{(r,\theta,z) \in \mathbb{R}^3 \mid 0 \leq r \leq 2, \, 0 \leq \theta < 2\pi, \, -2 \leq z \leq 3\}$ under the (polar) coordinate transformation $T : (r,\theta,z) \mapsto (r \cos(\theta), r \sin(\theta), z)$.
The Jacobian of this transformation is

\[ J(T) = \det \begin{vmatrix} \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \end{vmatrix} \]

\[ = \det \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} \]

\[ = r \left( \cos^2(\theta) + \sin^2(\theta) \right) \]

Thus,

\[ \int_D ze^{x^2+y^2} dA = \int_{D'} ze^{r^2} J(T) dA \]

\[ = \int_{-2}^{3} \int_{0}^{2\pi} \int_{0}^{2} ze^{r^2} r dr d\theta dz \]

\[ = 2\pi \int_{-2}^{3} \int_{0}^{2} ze^{r^2} dr dz \]

\[ = \pi \int_{-2}^{3} \int_{0}^{4} ze^u du dz \]

\[ = \pi \int_{-2}^{3} z \left( e^4 - 1 \right) dz \]

\[ = \frac{\pi}{2} \left( e^4 - 1 \right) \left( 3^2 - (-2)^2 \right) \]

\[ = \frac{5\pi}{2} \left( e^4 - 1 \right) \]