Section 4.2

4.2.1. Calculate the arc length of the following curves:

(a) \( \sigma(t) = (6t, 3t^2, t^3) \), \( t \in [0, 1] \)

- Well,
  \[ \sigma'(t) = (6, 6t, 3t^2) \]
  so,
  \[ \|\sigma'(t)\| = \sqrt{36 + 36t^2 + 9t^4} \]
  \[ = \sqrt{9(t^4 + 4t^2 + 4)} \]
  \[ = \sqrt{9(t^2 + 2)^2} \]
  \[ = 3(t^2 + 2) \]
  Thus,
  \[ L[\sigma] = \int_0^1 \|\sigma'(t)\| dt \]
  \[ = \int_0^1 3(t^2 + 2) dt \]
  \[ = \left[ t^3 + 6t \right]_0^1 \]
  \[ = \boxed{7} \]

(b) \( \sigma(t) = (\sin(3t), \cos(3t), 2t^{3/2}) \), \( t \in [0, 1] \)

- Well,
  \[ \sigma'(t) = \left( 3\cos(3t), -3\sin(3t), 3t^{1/2} \right) \]
  so,
  \[ \|\sigma'(t)\| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 9t} \]
  \[ = \sqrt{9(1 + t)} \]
  \[ = 3\sqrt{1 + t} \]
  Thus,
  \[ L[\sigma] = \int_0^1 \|\sigma'(t)\| dt \]
  \[ = \int_0^1 3\sqrt{1 + t} dt \]
  \[ = \left[ 3u^{3/2} \right]_0^1 \]
  \[ = 2^{3/2} - 1 \]
4.2.2. Let $\sigma$ be the path $\sigma(t) = (t, t \sin(t), t \cos(t))$. Find the arc length of $\sigma$ between $(0,0,0)$ and $(\pi, 0, -\pi)$.

- Well,

$$\sigma'(t) = (1, \sin(t) + t \cos(t), \cos(t) - t \sin(t))$$

so

$$\|\sigma'(t)\| = \sqrt{1^2 + (\sin(t) + t \cos(t))^2 + (\cos(t) - t \sin(t))^2}$$

$$= \sqrt{1 + \sin^2(t) + t^2 \cos^2(t) + \cos^2(t) + t^2 \sin^2(t)}$$

$$= \sqrt{2 + t^2}$$

Note also that we must have $t_i = 0$ and $t_f = \pi$ so that

$$\sigma(t_i) = (0, 0, 0)$$

$$\sigma(t_f) = (\pi, 0, -\pi)$$

Therefore, the arc length will be given by the following integral

$$L[\sigma] = \int_{t_i}^{t_f} \|\sigma'(t)\| \, dt$$

$$= \int_0^\pi \sqrt{2 + t^2} \, dt$$

$$= \frac{\pi}{2} \sqrt{\pi^2 + 2} + \log \left| \frac{\pi + \sqrt{\pi^2 + 2}}{\sqrt{2}} \right|$$

(See integral #43 in the tables at the back of the text.)

Section 4.3

4.3.1. A particle of mass $m$ moves along a path $\mathbf{r}(t)$ according to Newton’s law in a force field $\mathbf{F} = -\nabla V$ on $\mathbb{R}^3$, where $V$ is a given potential energy function.

(a) Prove that the energy along the trajectory

$$E = \frac{1}{2} m \|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))$$

is constant in time.

- We have

$$\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} m \|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t)) \right)$$

$$= \frac{m}{2} \frac{d}{dt} (\mathbf{r}'(t) \cdot \mathbf{r}'(t)) + \frac{d}{dt} (V(\mathbf{r}(t)))$$

$$= \frac{m}{2} (\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t)) + \nabla V \cdot \frac{d\mathbf{r}}{dt}$$

$$= m \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \nabla V \cdot \mathbf{r}'(t)$$

(In the third line we have simply applied the product and chain rules to, respectively, the first and second terms of the second line.) According to Newton’s law $\mathbf{F} = ma$, so

$$m \mathbf{r}'' = \mathbf{F} = -\nabla V$$

Thus,

$$\frac{dE}{dt} = -\nabla V \cdot \mathbf{r}'(t) + \nabla V \cdot \mathbf{r}'(t) = 0$$

(b) If the particle moves on an equipotential surface, show that its speed is constant.

- Well, the particle speed is just the magnitude of the velocity vector. So it suffices to prove that

$$\frac{d}{dt} (|r'(t)|^2) = 0$$

whenever the particle moves along an equipotential surface.

But

$$\frac{d}{dt} (|r'(t)|^2) = \frac{d}{dt} (r'(t) \cdot r'(t)) = r''(t) \cdot r'(t) + r'(t) \cdot r''(t) = 2r'(t) \cdot r''(t) = \frac{2}{m} r'(t) \cdot (mr''(t)) = \frac{2}{m} r'(t) \cdot \nabla V (r(t))$$

Now we know from Section 2.5, that the gradient vector $\nabla V$ evaluated at $r(t)$ will be normal to the surface

$$S = \{ x \in \mathbb{R}^3 \mid V(x) = k \}$$

at the point $r(t)$. On the other hand, since the trajectory is constrained to lie in such a surface, the tangent vector $r'(t)$ at a point $r(t)$ must always be perpendicular to the surface normal. In other words,

$$r'(t) \cdot \nabla V (r(t)) = 0 .$$

Thus,

$$\frac{d}{dt} (|r'(t)|^2) = -\frac{2}{m} r'(t) \cdot \nabla V (r(t)) = 0 .$$

4.3.2. Sketch a few flow lines of the vector field $F(x,y) = (x, -y)$.

- The flow lines for this vector field must satisfy the differential equation

$$\frac{d\sigma}{dt} = F(\sigma(t))$$

But

$$\left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \left( \begin{array}{c} \sigma_x(t) \\ -\sigma_y(t) \end{array} \right) \Rightarrow \frac{dx}{dt} = \sigma_x \Rightarrow \sigma_x(t) = x_0 e^t$$

so the flow lines of $F$ will be curves of the form

$$\sigma(t) = (x_0 e^t, y_0 e^{-t}) .$$
4.3.3. Let \( c(t) \) be a flow line of a gradient field \( F = -\nabla V \). Prove that \( V(c(t)) \) is a decreasing function of \( t \). Explain.

\[
\begin{align*}
\frac{d}{dt} [V(c(t))] &= \nabla V(c(t)) \cdot \frac{dc}{dt} \\
&= \nabla V(c(t)) \cdot F(c(t)) \\
&= \nabla V(c(t)) \cdot (-\nabla V(c(t))) \\
&= -\|\nabla V(c(t))\|^2
\end{align*}
\]

Since the magnitude of a vector is either positive or zero, we conclude that \( \frac{d}{dt} [V(c(t))] \) is either negative or zero.

To understand this, recall that \( -\nabla V(r) \) represents the direction of the fastest decrease in \( V \) at the point \( r \). Thus, the flow lines of a vector field \( F = -\nabla V \) will always move in the direction of the fastest decrease in \( V \); \( V \) obviously \( V \) will be decreasing along these flow lines.

In a physical situation, \( F \) is interpretable as a force field and \( V \) is a corresponding potential energy. The fact that \( V \) is always decreasing along the flow lines of \( F = -\nabla V \) implies that a particle acted upon by \( F \) always moves along a path that decreases its potential energy. (Now you know why apples fall.)

4.3.4. Sketch the gradient field \(-\nabla V\) for \( V(x,y) = \frac{x+y}{x^2+y^2} \). Sketch the equipotential surface \( V = 1 \).
- The easiest way to approach this problem is first uncover the nature of the equipotential surfaces. Now the points on an equipotential surface for $V$ must satisfy an equation of the form

$$\frac{x + y}{x^2 + y^2} = k$$

which is equivalent to

$$x^2 - \frac{1}{k}x + y^2 - \frac{1}{k}y = 0$$

which, upon adding $2 \left( \frac{1}{2k} \right)^2$ to both sides, becomes

$$x^2 - \frac{1}{k}x + \left( \frac{1}{2k} \right)^2 + y^2 - \frac{1}{k}y + \left( \frac{1}{2k} \right)^2 = 2 \left( \frac{1}{2k} \right)^2$$

or

$$\left(x - \frac{1}{2k}\right)^2 + \left(y - \frac{1}{2k}\right)^2 = 1 \frac{1}{2k^2}.$$ 

This is the equation of a circle of radius $\frac{1}{\sqrt{2k^2}}$ centered about the point $\left( \frac{1}{2k}, \frac{1}{2k} \right)$. Noting that the distance of the point $\left( \frac{1}{2k}, \frac{1}{2k} \right)$ from the origin is precisely $\frac{1}{\sqrt{2k^2}}$, we can conclude that equipotential surfaces are circles that always contain the origin $(0,0)$, and whose their centers will lie along the line $x = y$.

The flow lines of the gradient field $\mathbf{F} = -\nabla V$ will always be anti-parallel to $\nabla V$ which will always be perpendicular to the equipotential surfaces (this we know from Section 2.5). Thus, to sketch the vector field $\mathbf{F}$ we can sketch the equipotential surfaces and then draw vectors that are perpendicular to the equipotential surfaces.
4.3.5. Show that \( \sigma(t) = (e^{2t}, \ln |t|, 1/t) \) for \( t \neq 0 \) is a flow line of the velocity vector field \( \mathbf{F}(x, y, z) = (2x, z, -z^2) \).

- Well,

\[
\frac{d\sigma_x}{dt}(t) = 2e^{2t} = 2\sigma_x(t) = F_x(\sigma(t))
\]
\[
\frac{d\sigma_y}{dt}(t) = \frac{1}{t} = \sigma_z(t) = F_y(\sigma(t))
\]
\[
\frac{d\sigma_z}{dt} = -\frac{1}{t^2} = - (\sigma_z(t))^2 = F_z(\sigma(t))
\]

Thus

\[
\frac{d\sigma}{dt}(t) = \mathbf{F}(\sigma(t))
\]

and so \( \sigma(t) \) is a flow line of \( \mathbf{F} \).

**Section 4.4**

4.4.1. Compute the curl, \( \nabla \times \mathbf{F} \), of each of the following vector fields.

(a) \( \mathbf{F}(x, y, z) = xi + yj + zk \)

- We have

\[
\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}
\]

\[
= (0, -0, 0 - 0)
\]

\[
= (0, 0, 0)
\]

(b) \( \mathbf{F}(x, y, z) = yzi + xzj + xyk \)
We have
\[ \nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]
\[ = (x - x, y - y, z - z) \]
\[ = (0, 0, 0) \]

(c) \( \mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)^3 (31 + 4\mathbf{j} + 5\mathbf{k}) \)

\[ \nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]
\[ = (10y - 8z, 6z - 10x, 8x - 6y) \]

4.4.2. Compute the divergence of each of the vector fields in Exercise 1.

(a)
\[ \nabla \cdot \mathbf{F} = \nabla \cdot (x, y, z) \]
\[ = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \]
\[ = 1 + 1 + 1 \]
\[ = 3 \]

(b)
\[ \nabla \cdot \mathbf{F} = \nabla \cdot (yz, xz, xy) \]
\[ = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) \]
\[ = 0 + 0 + 0 \]
\[ = 0 \]

(c)
\[ \nabla \cdot \mathbf{F} = \nabla \cdot (3x^2 + 3y^2 + 3z^2, 4x^2 + 4y^2 + 4z^2, 5x^2 + 5y^2 + 5z^2) \]
\[ = \frac{\partial}{\partial x}(3x^2 + 3y^2 + 3z^2) + \frac{\partial}{\partial y}(4x^2 + 4y^2 + 4z^2) + \frac{\partial}{\partial z}(5x^2 + 5y^2 + 5z^2) \]
\[ = 6x + 8y + 10z \]

4.4.3. Let \( \mathbf{F}(x, y, z) = 3x^2 y\mathbf{i} + (x^3 + y^3)\mathbf{j} \).

(a) Verify that \( \nabla \times \mathbf{F} = 0 \).
\[ \nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 - 0 - 0, 3x^2 - 3x^2 \\ 0 - 0 - 0, 3x^2 - 3x^2 \\ 0 - 0 - 0, 3x^2 - 3x^2 \end{pmatrix} = (0, 0, 0) \]

(b) Find a function \( f \) such that \( \mathbf{F} = \nabla f \).

- We need to find a function \( f : \mathbb{R}^3 \to \mathbb{R} \) such that

\[
\frac{\partial f}{\partial x} = 3x^2y \\
\frac{\partial f}{\partial y} = x^3 + y^3 \\
\frac{\partial f}{\partial z} = 0
\]

Now the most general function \( f \) of \( x, y, z \) satisfying the first equation in (B1) will be of the form

\[
f(x, y, z) = \int 3x^2y \, dx + h_1(y, z) = x^3y + h_1(y, z) \quad (B2)
\]

Here \( h_1(y, z) \) is an arbitrary function of \( y \) and \( z \).

The most general function satisfying the second equation in (B2) will be of the form

\[
f(x, y, z) = \int (x^3 + y^3) \, dy + h_2(x, z) = x^3y + \frac{1}{4}y^4 + h_2(x, z) \quad (B3)
\]

where \( h_2(x, z) \) is an arbitrary function of \( x \) and \( z \).

The most general function satisfying the third equation (B3) will be of the form

\[
f(x, y, z) = \int 0 \cdot dz + h_3(x, y) = h_3(x, y) \quad (B4)
\]

Now the function \( f \) that we seek must satisfy (B2), (B3), and (B4) simultaneously. Equation (B2) tells us that the \( x \) dependence of \( f \) lies solely in a term of the form \( x^3y \); equation (B3) tells us that the \( y \) dependence of \( f \) lies solely in the sum of two terms \( x^3y + \frac{1}{4}y^4 \); and equation (B4) tells us that \( f \) does not depend at all on \( z \). We can thus conclude that any function of the form

\[
f(x, y, z) = x^3y + \frac{1}{4}y^4 + C
\]

will be a solution of \( \nabla f = F \).

(c) Is it true that for a vector field \( \mathbf{F} \) such a function can exist only if \( \nabla \times \mathbf{F} = 0 \)?

- Suppose \( \mathbf{F} = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \). Then

\[
\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial y} \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \frac{\partial f}{\partial y} \\ \frac{\partial}{\partial z} \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z} \\ \frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \end{pmatrix}
\]

Now by Theorem 15 (Section 2.6), if \( f \) is of class \( C^2 \), then

\[
0 = \frac{\partial}{\partial y} \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \frac{\partial f}{\partial y} = \frac{\partial}{\partial z} \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \frac{\partial f}{\partial x}
\]

We conclude that if \( \nabla \times \mathbf{F} \neq 0 \), there can be no function of class \( C^2 \) such that \( \mathbf{F} = \nabla f \). \( \blacksquare \)

4.4.4. Show that \( \mathbf{F} = y(\cos(x))\mathbf{i} + x(\sin(y))\mathbf{j} \) is not a gradient field.
• Suppose that $\mathbf{F} = \nabla f$. Then

$$\frac{\partial f}{\partial x} = y \cos(x)$$

$$\frac{\partial f}{\partial y} = x \sin(y)$$

Each of the two functions on the right hand side are perfectly continuous, and moreover, their partial derivatives exist and are continuous for all $x$ and $y$. Therefore, $f$ is at least of class $C^2$. But then, by Theorem 15 of Section 2.6, we must have

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}.$$

But

$$\frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \cos(x) \neq \sin(x) = \frac{\partial}{\partial x} \frac{\partial f}{\partial y}.$$

We conclude that $\mathbf{F}$ can not be a gradient field. 

Section 4.5

4.5.1. Suppose $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following vector fields necessarily have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$

• By Identity 5 on page 283 we have

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} = 0 + 0 = 0.$$

(b) $\mathbf{F} \times \mathbf{G}$

• By Identity 9 on page 283 we have

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

The expression of the right hand side does not necessarily vanish (even if $0 = \nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}$). For example, if

$$\mathbf{F} = (-y, x, 0)$$

$$\mathbf{G} = (0, 0, 1)$$

Then

$$0 = \nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}$$

and

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

$$= (0, 0, 1) \cdot (0, 0, 2) - (-y, x, 0) \cdot (0, 0, 0)$$

$$= 2.$$

(c) $(\mathbf{F} \cdot \mathbf{G}) \mathbf{F}$
By Identities 8 and 7 on page 283 we have

\[
\nabla \cdot ((\mathbf{F} \cdot \mathbf{G}) \mathbf{F}) = (\mathbf{F} \cdot \mathbf{G}) (\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla (\mathbf{F} \cdot \mathbf{G})
\]

\[
= (\mathbf{F} \cdot \mathbf{G}) (\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot (\nabla \cdot (\mathbf{F} \cdot \mathbf{G})) + \mathbf{F} \cdot (\mathbf{G} \cdot (\nabla \times \mathbf{F}))
\]

\[
= 0 + \mathbf{F} \cdot ((\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}))
\]

The expression of the right hand side does not necessarily vanish (even if \(0 = \nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}\)).

4.5.2. Prove the following identities.

(a) \(\nabla (\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})\)

By virtue of the product rule the left hand side is equivalent to

\[
LHS = \nabla (\mathbf{F} \cdot \mathbf{G}) =
\]

\[
= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) (F_x G_z + F_z G_y + F_z G_z)
\]

\[
= \left(\frac{\partial F_x}{\partial x} G_z + F_x \frac{\partial G_z}{\partial x} + \frac{\partial F_y}{\partial y} G_z + F_y \frac{\partial G_z}{\partial y} + \frac{\partial F_z}{\partial z} G_z + F_z \frac{\partial G_z}{\partial z}\right) i
\]

\[
+ \left(\frac{\partial F_x}{\partial y} G_y + F_x \frac{\partial G_y}{\partial y} + \frac{\partial F_y}{\partial y} G_y + F_y \frac{\partial G_y}{\partial y} + \frac{\partial F_z}{\partial z} G_y + F_z \frac{\partial G_y}{\partial z}\right) j
\]

\[
+ \left(\frac{\partial F_x}{\partial z} G_z + F_x \frac{\partial G_z}{\partial z} + \frac{\partial F_y}{\partial z} G_z + F_y \frac{\partial G_z}{\partial z} + \frac{\partial F_z}{\partial z} G_z + F_z \frac{\partial G_z}{\partial z}\right) k
\]

On the other hand,

\[
(\mathbf{F} \cdot \nabla) \mathbf{G} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) (G_x, G_y, G_z)
\]

\[
= \left(\frac{\partial G_x}{\partial x} + G_y \frac{\partial F_x}{\partial y} + G_z \frac{\partial F_x}{\partial z}\right) i
\]

\[
+ \left(\frac{\partial G_y}{\partial y} + G_y \frac{\partial F_y}{\partial y} + G_z \frac{\partial F_y}{\partial z}\right) j
\]

\[
+ \left(\frac{\partial G_z}{\partial z} + G_y \frac{\partial F_z}{\partial z} + G_z \frac{\partial F_z}{\partial z}\right) k
\]

\[
(\mathbf{G} \cdot \nabla) \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) (F_x, F_y, F_z)
\]

\[
= \left(\frac{\partial F_x}{\partial x} + G_y \frac{\partial F_x}{\partial y} + G_z \frac{\partial F_x}{\partial z}\right) i
\]

\[
+ \left(\frac{\partial F_y}{\partial y} + G_y \frac{\partial F_y}{\partial y} + G_z \frac{\partial F_y}{\partial z}\right) j
\]

\[
+ \left(\frac{\partial F_z}{\partial z} + G_y \frac{\partial F_z}{\partial z} + G_z \frac{\partial F_z}{\partial z}\right) k
\]
\[
\mathbf{F} \times (\nabla \times \mathbf{G}) = (F_z, F_y, F_x) \times \left( \begin{array}{c}
\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \\
\frac{\partial G_z}{\partial z} - \frac{\partial G_x}{\partial z} \\
\frac{\partial G_x}{\partial x} - \frac{\partial G_z}{\partial x}
\end{array} \right)
\]

\[
= \left( F_y \frac{\partial G_z}{\partial y} - F_y \frac{\partial G_y}{\partial z} - F_z \frac{\partial G_z}{\partial x} + F_z \frac{\partial G_z}{\partial x} \right) \mathbf{i} \\
+ \left( F_z \frac{\partial G_z}{\partial x} - F_z \frac{\partial G_y}{\partial z} - F_x \frac{\partial G_y}{\partial y} + F_x \frac{\partial G_x}{\partial y} \right) \mathbf{j} \\
+ \left( F_x \frac{\partial G_x}{\partial y} - F_x \frac{\partial G_y}{\partial y} + F_y \frac{\partial G_y}{\partial z} + F_y \frac{\partial G_y}{\partial z} \right) \mathbf{k}
\]

\[
\mathbf{G} \times (\nabla \times \mathbf{F}) = (G_z, G_y, G_x) \times \left( \begin{array}{c}
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\
\frac{\partial F_z}{\partial z} - \frac{\partial F_x}{\partial z} \\
\frac{\partial F_x}{\partial x} - \frac{\partial F_z}{\partial x}
\end{array} \right)
\]

\[
= \left( G_y \frac{\partial F_z}{\partial y} - G_y \frac{\partial F_y}{\partial z} - G_z \frac{\partial F_z}{\partial x} + G_z \frac{\partial F_z}{\partial x} \right) \mathbf{i} \\
+ \left( G_z \frac{\partial F_z}{\partial x} - G_z \frac{\partial F_y}{\partial z} - G_y \frac{\partial F_y}{\partial y} + G_y \frac{\partial F_x}{\partial y} \right) \mathbf{j} \\
+ \left( G_y \frac{\partial F_y}{\partial z} - G_y \frac{\partial F_x}{\partial y} + G_x \frac{\partial F_x}{\partial y} + G_x \frac{\partial F_x}{\partial y} \right) \mathbf{k}
\]

And so the right hand side of (a) is

\[
RHS = (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})
\]

\[
= \left( \frac{\partial F_x}{\partial x} G_x - \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z + \frac{\partial F_x}{\partial x} G_x + \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z \right) \mathbf{i} \\
+ \left( \frac{\partial F_x}{\partial x} G_x - \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z + \frac{\partial F_x}{\partial x} G_x + \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z \right) \mathbf{j} \\
+ \left( \frac{\partial F_x}{\partial x} G_x - \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z + \frac{\partial F_x}{\partial x} G_x + \frac{\partial F_y}{\partial y} G_y + \frac{\partial F_z}{\partial z} G_z \right) \mathbf{k}
\]

which is equivalent to the left hand side of identity (a). \( \blacksquare \)

\[
(b) \ \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\]
\[ \nabla \cdot (F \times G) = (G_z \frac{\partial F_y}{\partial z} - G_z \frac{\partial F_x}{\partial z} + G_y \frac{\partial F_z}{\partial z} + G_y \frac{\partial F_x}{\partial z} - G_z \frac{\partial F_y}{\partial z} - G_z \frac{\partial F_x}{\partial z}) \]

\[ G \cdot (\nabla \times F) = (G_z, G_y, G_z) \times (F_x \frac{\partial G_y}{\partial z} - F_x \frac{\partial G_z}{\partial z} + F_y \frac{\partial G_z}{\partial z} - F_y \frac{\partial G_x}{\partial z} + F_z \frac{\partial G_x}{\partial z} - F_z \frac{\partial G_y}{\partial z}) \]

\[ F \cdot (\nabla \times G) = (F_x, F_y, F_z) \times (G_x \frac{\partial F_y}{\partial z} - G_x \frac{\partial F_z}{\partial z} + G_y \frac{\partial F_z}{\partial z} - G_y \frac{\partial F_x}{\partial z} + G_z \frac{\partial F_x}{\partial z} - G_z \frac{\partial F_y}{\partial z}) \]

So

\[ G \cdot (\nabla \times F) - F \cdot (\nabla \times G) = \nabla \cdot (F \cdot G) \]
We have
\[ \nabla \times (f F_x, f F_y, f F_z) = \begin{pmatrix} F_z \partial f \partial y + f \partial F_z \partial y - F_y \partial f \partial z - f \partial F_y \partial z \\ F_z \partial f \partial z + f \partial F_z \partial z - F_y \partial f \partial x - f \partial F_y \partial x \\ F_y \partial f \partial x + f \partial F_y \partial x - F_x \partial f \partial y - f \partial F_x \partial y \end{pmatrix} i + \begin{pmatrix} F_z \partial f \partial y + f \partial F_z \partial y - F_y \partial f \partial z - f \partial F_y \partial z \\ F_z \partial f \partial z + f \partial F_z \partial z - F_y \partial f \partial x - f \partial F_y \partial x \\ F_y \partial f \partial x + f \partial F_y \partial x - F_x \partial f \partial y - f \partial F_x \partial y \end{pmatrix} j + \begin{pmatrix} F_z \partial f \partial y + f \partial F_z \partial y - F_y \partial f \partial z - f \partial F_y \partial z \\ F_z \partial f \partial z + f \partial F_z \partial z - F_y \partial f \partial x - f \partial F_y \partial x \\ F_y \partial f \partial x + f \partial F_y \partial x - F_x \partial f \partial y - f \partial F_x \partial y \end{pmatrix} k \]

\[ = f \left( \begin{pmatrix} \partial F_z \partial y - \partial F_y \partial z \\ \partial F_x \partial z - \partial F_z \partial x \\ \partial F_y \partial x - \partial F_x \partial y \end{pmatrix} + \nabla f \times \mathbf{F} \right) \]

4.5.3. Let \( \mathbf{F} = (2xz^2, 1, y^3 z x) \), \( \mathbf{G} = (x^2, y^2, z^3) \), and \( f = x^2 y \). Compute the following quantities.

(a) \( \nabla f \)

\[ \nabla f = (2xy, x^2, 0) \]

(b) \( \nabla \times \mathbf{F} \)

\[ \nabla \times \mathbf{F} = (3y^2 z x, 4xz - y^3 z, 0) \]

(c) \( (\mathbf{F} \cdot \nabla) \mathbf{G} \)

\[ (\mathbf{F} \cdot \nabla) \mathbf{G} = \begin{pmatrix} 2xz^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + y^3 z x \frac{\partial}{\partial z} \end{pmatrix} \cdot (x^2, y^2, z^3) \]

\[ = (4x^2 z^2, 2y, 2y^3 z^2 x) \]

(d) \( \mathbf{F} \cdot (\nabla f) \)

\[ \mathbf{F} \cdot (\nabla f) = (2xz^2, 1, y^3 z x) \cdot (2xy, x^2, 0) \]

\[ = 4x^2 y z^2 + x^2 \]

(e) \( \mathbf{F} \times \nabla f \)

\[ \mathbf{F} \times (\nabla f) = (2xz^2, 1, y^3 z x) \times (2xy, x^2, 0) \]

\[ = (-y^3 z x^3, 2y^4 x^2 z, 2x^3 z^2 - 2xy) \]
4.5.4. Let \( \mathbf{F} \) be a general vector field. Does \( \nabla \times \mathbf{F} \) have to be perpendicular to \( \mathbf{F} \)?

- No, consider the vector field

\[
\mathbf{F}(x, y, z) = (-y, x, 1).
\]

We have

\[
\nabla \times \mathbf{F} = (0 - 0, 0 - (-1), 0) = (0, 0, 2).
\]

So,

\[
\mathbf{F} \cdot (\nabla \times \mathbf{F}) = (-y, x, 1) \cdot (0, 0, 2) = 2 \neq 0.
\]

Thus, \( \nabla \times \mathbf{F} \) is not perpendicular to \( \mathbf{F} \).