Integrals over 3-Dimensional Regions

1. Integrals over Rectangular Boxes

The definition of an integral over a 3-dimensional rectangular box is a straight-forward generalization of the definition of an integral over a (2-dimensional) rectangle.

**Definition 19.1.** A Riemann sum of a function \( f : \mathbb{R}^3 \to \mathbb{R} \) over a rectangular box

\[
R = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}
\]

is a sum of the form

\[
S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(p_{ijk}) \phi \Delta x \Delta y \Delta z
\]

where

\[
\Delta x = \frac{b - a}{n}, \quad \Delta y = \frac{d - c}{n}, \quad \Delta z = \frac{f - e}{n}
\]

and \( p_{ijk} \) is a point within the rectangular box

\[
R_{ijk} = \{(x, y, z) \in \mathbb{R}^3 \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j, z_{k-1} \leq z \leq z_k\}
\]

where

\[
x_i = a + i \Delta x, \quad y_j = c + j \Delta y, \quad z_k = e + k \Delta z
\]

**Definition 19.2.** The integral of a function \( f : \mathbb{R}^3 \to \mathbb{R} \) over a rectangular box \( R \) is the limit of a sequence of Riemann sums of \( f \) over \( R \)

\[
\int_R f(x, y, z) dV = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(p_{ijk}) \phi \Delta x \Delta y \Delta z
\]

whenever this limit exists and is independent of the choice of points \( p_{ijk} \).

**Theorem 19.3.** If \( f : \mathbb{R}^3 \to \mathbb{R} \) is continuous on \( R \) then

\[
\int_R f(x, y, z) dV
\]

exists and

\[
\int_R f(x, y, z) dV = \int_a^b \left( \int_c^d \left( \int_e^f f(x, y, z) dz \right) dy \right) dx
\]

Moreover, its value is independent of the order of integration on the right hand side.
Example 19.4. Evaluate
\[
\int_R xyz \, dV
\]
where \( R \) is the rectangular box
\[
R = \{ (x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1 \}
\]
- We have
\[
\int_R xyz \, dV = \int_{-1}^{1} \left( \int_{0}^{2} \left( \int_{0}^{1} xyz \, dz \right) \, dy \right) \, dx = \int_{-1}^{1} \left( \int_{0}^{2} \left( \frac{1}{2}yx - 0 \right) \, dy \right) \, dx = \int_{-1}^{1} \left( \frac{1}{4}x(2^2) - 0 \right) \, dx = \frac{1}{2} (1)^2 - \frac{1}{2} (-1)^2 = 0
\]

2. Integrals over More General Regions

Definition 19.5. By an elementary region in \( \mathbb{R}^3 \) we shall mean a region that can be prescribed by
- restricting one coordinate, say \( x_3 \), to lie between the graphs of two functions of the other coordinates
  \( \psi_1 (x_1, x_2) \leq x_3 \leq \psi_2 (x_1, x_2) \)
- restricting a second coordinate, say \( x_2 \), to lie between the graphs of two functions of the remaining coordinate
  \( \phi_1 (x_1) \leq x_2 \leq \phi_2 (x_1) \)
- restricting the last coordinate to lie between two constants
  \( a \leq x_1 \leq b \)

Theorem 19.6. Suppose \( S \) is an elementary region in \( \mathbb{R}^3 \) and \( f(x, y, z) \) is continuous on \( S \). Then
\[
\int_S f(x, y, z) \, dV = \int_a^b \left( \int_{\phi_2(x_1)}^{\phi_1(x_1)} \left( \int_{\psi_2(x_1, x_2)}^{\psi_1(x_1, x_2)} f(x_1, x_2, x_3) \, dx_3 \right) \, dx_2 \right) \, dx_1
\]

Example 19.7. Let \( B \) be a ball of radius 1 centered at the origin. Compute
\[
\int_B dV
\]
- We can realize the ball as an elementary region as follows.
\[
B = \{ -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2} \}
\]
So
\[
\int_B dV = \int_{-1}^{1} \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left( \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz \right) \, dy \right) \, dx = \int_{-1}^{1} \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left( 2 \sqrt{1-x^2-y^2} \right) \, dy \right) \, dx
\]
Using the identity

\[ \int_{-a}^{a} \sqrt{a^2 - y^2} dy = \frac{a^2}{2} \pi \]

we have

\[ \int_B dV = \int_{-1}^{1} 2 \left( \frac{1 - x^2}{2} \pi \right) \]

\[ = \pi \left( x - \frac{x^3}{3} \right) \bigg|_{-1}^{1} \]

\[ = \pi \left( 1 - \frac{1}{3} - \left( -1 + \frac{1}{3} \right) \right) \]

\[ = \frac{4}{3} \pi \]