1. Equations of Lines and Planes

1.1. Equation of a Line in \( \mathbb{R}^3 \). There are two common geometrical ways of describing a straight line in a 3-dimensional space.

- Given two distinct points \( p_1, p_2 \in \mathbb{R}^3 \), there is a unique line passing through both \( p_1 \) and \( p_2 \).
- Given one point \( p \in \mathbb{R}^3 \) and a direction \( v \), there is a unique line passing through \( p_0 \) with the direction \( v \).

In this course, we shall think of a lines sets of points of the following form

\[
L = \{ p \in \mathbb{R}^3 \mid p = p_0 + vt, \quad t \in \mathbb{R} \}
\] (2.1)

The connection with the second geometrical description of a line is evident from the notation. To make the connection with the first geometrical description, all we have to do is set \( p_0 = p_1 \) and set \( v = p_2 - p_1 \).

If we express the vectors \( p, p_0, \) and \( v \) in terms of components; e.g.

\[
p = (x, y, z) \\
p_0 = (x_0, y_0, z_0) \\
v = (v_x, v_y, v_z)
\]

the we obtain from (2.2) the following parametric equation for a line

\[
x = x_0 + v_x t \\
y = y_0 + v_y t \\
z = z_0 + v_z t
\]

In terms of the components of two points \( p_1 = (x_1, y_1, z_1) \) and \( p_2 = (x_2, y_2, z_2) \) lying on the line we have corresponding to the first geometrical description of a line the following parametric equation

\[
x = x_1 + (x_2 - x_1) t \\
y = y_1 + (y_2 - y_1) t \\
z = z_1 + (z_2 - z_1) t
\]

1.2. Equation of a Plane in \( \mathbb{R}^3 \). Just as a line can be prescribed by specifying its direction and a single point on the line; a plane can be prescribed by specifying a single point \( p_0 \) lying in the plane and two distinct directions \( v, u \) lying in the plane. In vector notation such a prescription takes the form

\[
P = \{ p \in \mathbb{R}^3 \mid p = p_0 + us + vt, \quad s, t \in \mathbb{R} \}
\]
If we set
\[ p = (x, y, z) \]
\[ u = (u_x, u_y, u_z) \]
\[ v = (v_x, v_y, v_z) \]
then the relation \( p = p_0 + us + vt \) leads to the following parametric representation of a plane
\[ x = x_0 + u_x s + v_x t \]
\[ y = y_0 + u_y s + v_y t \]
\[ z = z_0 + u_z s + v_z t \]

Another way of prescribing a plane is to specify one point \( p_0 \) lying in the plane and the direction of a vector \( n \) that is perpendicular to the plane. If another point \( p_1 \) is to lie in the plane the plane, the vector from \( p_0 \) to \( p_1 \) must be perpendicular to \( n \), since \( n \) is perpendicular to every direction in the plane. In terms of vector notation we must have
\[ 0 = n \cdot (p_1 - p_0) \]
If we set
\[ n = (n_x, n_y, n_z) \]
\[ p_1 = (x, y, z) \]
\[ p_0 = (x_0, y_0, z_0) \]
then we have
\[ 0 = n \cdot (p_1 - p_0) = n_x (x - x_0) + n_y (y - y_0) + n_z (z - z_0) \]

1.3. Applications.

**Example 2.1.** Find the line passing through the point \((3, 1, -2)\) that intersects the line \( l_0 \) perpendicularly.

\[ x = -1 + t \]
\[ y = -2 + t \]
\[ z = -1 + t \]

perpendicularly.

- The vector equation for the line \( l_0 \) is
\[ l_0 = (-1, -2, -1) + t(1, 1, 1) \]
ans so the direction of the line \( l_0 \) is \( v_0 = (1, 1, 1) \). If \( l \) is a line through the point \((3, 1, -2)\) then it has an equation of the form
\[ l = (3, 1, -2) + tv \]
Now if \( l \) intersects \( l_0 \) perpendicularly the direction \( v \) of \( l \) must be perpendicular to the direction \((1, 1, 1)\) of \( l_0 \). Therefore
\[ 0 = v \cdot (1, 1, 1) \]
\[ = v_x + v_y + v_z \]
We know also have a point \((x, y, z)\) common to both lines so
\[ -1 + t = x = 3 + v_x s \]
\[ -2 + t = y = 1 + v_y s \]
\[ -1 + t = z = -2 + v_z s \]
Note that we can set \( s = 1 \) if we simultaneously rescale the direction vector \( \mathbf{v} \). We thus arrive at four equations for four unknowns

\[
\begin{align*}
v_x + v_y + v_z &= 0 \\
v_x - t &= -4 \\
v_y - t &= -3 \\
v_z - t &= 1
\end{align*}
\]

If we sum the last three equations we get
\[
v_x + v_y + v_z - 3t = -6
\]
or, using the first equation,
\[
-3t = -6 \quad \Rightarrow \quad t = 2
\]

We then find
\[
\begin{align*}
v_x &= -4 + t = -2 \\
v_y &= -3 + t = -1 \\
v_z &= 1 + t = 3
\end{align*}
\]

Thus, \( \mathbf{v} = (-2, -1, 3) \) and the equation of the line \( l \) is
\[
l = (3, 1, -2) + t(-2, -1, 3)
\]

**Example 2.2.** Find the equation of the plane that contains the point \((2, -1, 3)\) and is perpendicular to the line
\[
l = (1, -1, 3) + t(3, -2, 4)
\]

- If \( \mathbf{p} = (x, y, z) \) is a point on this plane, then line from the point \( \mathbf{p}_0 = (2, -1, 3) \) to \( \mathbf{p} \) will also lie in the plane and so must be perpendicular to the direction \( \mathbf{v} = (3, -2, 4) \) of \( l \). This leads to the condition

\[
0 = \mathbf{v} \cdot (\mathbf{p} - \mathbf{p}_0) \\
= (3, -2, 4) \cdot (x - 2, y - 1, z - 3) \\
= 3x - 6 - 2y + 2 + 4z - 12 \\
= 3x - 2y + 4z - 16
\]

The equation of the plane is thus
\[
3x - 2y + 4z = 16
\]

**Example 2.3.** Find the equation of the plane containing the lines \( l_1 = (0, 1, 1) + t(1, 2, 1) \) and \( l_2 = (0, 1, 0) + t(1, -1, 1) \)

- The direction of the first line is \( \mathbf{v}_1 = (1, 2, 1) \), the direction of the second line is \( \mathbf{v}_2 = (1, -1, 1) \), and the direction that is perpendicular to both these lines is

\[
\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 \\
= (1, 2, 1) \times (1, -1, 1) \\
= ((2)(1) - (1)(-1), (1)(1) - (1)(1), (1)(-1) - (2)(1)) \\
= (3, 0, 3)
\]

Every other vector in the plane must also be perpendicular to \( \mathbf{n} \).
Since the point $(0,1,1)$ lies in the line $l_1$, which in turn lies in the plane, $p_0 = (0,1,1)$ is a point lying in the plane. If $p = (x,y,z)$ is any other point in the plane, then the displacement vector $p - p_0 = (x,y-1,z-1)$ must also lie in the plane and must be perpendicular to $n$. Therefore

$$0 = n \cdot (p - p_0)$$
$$= (3,0,3) \cdot (x,y-1,z-1)$$
$$= 3x + 3z - 3$$

The equation of the plane is thus

$$3x + 3z = 3$$