Section 3.1

3.1.1. Compute the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$ for each of the following functions. Verify Theorem 15 in each case.

(a) $f(x, y) = \frac{2xy}{(x^2 + y^2)^2}$, $(x, y) \neq 0$.

(b) $f(x, y, z) = e^z + \frac{1}{x} + xe^{-y}$, $x \neq 0$.

3.1.2. Let

$$f(x, y) = \begin{cases} xy \frac{(x^2 - y^2)}{(x^2 + y^2)} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = 0 \end{cases}$$

(a) If $(x, y) \neq 0$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Show that $\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0 = \left. \frac{\partial f}{\partial y} \right|_{(0,0)}$.

(c) Show that $\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = 1$, $\left. \frac{\partial^2 f}{\partial y \partial x} \right|_{(0,0)} = -1$.

3.1.3. A function $u = f(x, y)$ with continuous second partial derivatives satisfying Laplace’s equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a **harmonic function**. Show that $u(x, y) = x^3 - 3xy^2$ is harmonic.

Section 3.2

3.2.1. Determine the second order Taylor formula for $f(x, y) = (x + y)^2$ about $(0,0)$.

3.2.2. Determine the second order Taylor formula for $f(x, y) = \frac{1}{x^2 + y^2 + 1}$ about $(0,0)$.

3.2.3. Determine the second order Taylor formula for $f(x, y) = e^{x+y}$ about $(0,0)$.

3.2.4. Determine the second order Taylor formula for $f(x, y) = \sin(xy) + \cos(xy)$ about $(0,0)$.

Section 3.3

3.3.1. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = x^2 - y^2 + xy$$

3.3.2. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = x^2 + y^2 + 2xy$$
3.3.3. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

\[ f(x, y) = e^{1 + x^2 - y^2} \]

3.3.4. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

\[ f(x, y) = 3x^2 + 2xy + 2x + y^2 + y + 4 \]

3.3.5. An examination of the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto (y - 3x^2)(y - x^2) \) will give an idea of the difficulty of finding conditions that guarantee that a critical point is a relative extremum when Theorem 5 fails. Show that

(a) the origin is a critical point of \( f \);

(b) \( f \) has a relative minimum at (0,0) on every straight line through (0,0); that is, if \( g(t) = (at, bt) \), then \( f \circ g : \mathbb{R} \rightarrow \mathbb{R} \) has a relative minimum at 0, for every choice of \( a \) and \( b \);

(c) The origin is not a relative minimum of \( f \).

3.3.6. Let \( f(x, y) = x^2 - 2xy + y^2 \). Here \( D = 0 \). Can you say whether the critical points are local minima, local maxima, or saddle points?

Section 4.3

3.4.1. Find the extrema of \( f(x, y, z) = x - y + z \) subject to the constraint \( x^2 + y^2 + z^2 = 2 \).

3.4.2. Find the extrema of \( f(x, y) = x \) subject to the constraint \( x^2 + 2y^2 = 3 \).

3.4.3. Find the extrema of \( f(x, y) = 3x + 2y \) subject to the constraint \( 2x^2 + 3y^2 = 3 \).

Section 3.5

3.5.1.* Let \( F(x, y) = 0 \) define a curve in the \( xy \) plane through the point \((x_o, y_o)\). Assume that \((\partial F/\partial y)(x_o, y_o) \neq 0\). Show that this curve can be locally represented by the graph of a function \( y = g(x) \). Show that the line orthogonal to \( \nabla F(x_o, y_o) \) agrees with the tangent line to the graph of \( y = g(x) \).

4.3.5.2.* (a) Check directly (i.e., without using Theorem 10) where we can solve \( F(x, y) = y^2 + y + 3x + 1 = 0 \) for \( y \) in terms of \( x \).

(b) Check that your answer in part (a) agrees with the answer you expect from the implicit function theorem. Compute \( dy/dx \).

3.5.3.* Show that \( x^3z^2 - z^3y^2 = 0 \) is solvable for \( z \) as a function of \((x, y)\) near \((1,1,1)\), but not near the origin. Compute \( \partial z/\partial x \) and \( \partial z/\partial y \) at \((1,1)\).