LECTURE 11

Review Session for Second Midterm

I. Formal Definitions

A. Dimension
The dimension of a subspace $W$ is the number of vectors in any basis for $W$.

B. Row Space
The row space of an $n \times m$ matrix $A$ is subspace of $\mathbb{R}^m$ corresponding to the span of the row vectors of $A$.

C. Column Space
The column space of an $n \times m$ matrix $A$ is the subspace of $\mathbb{R}^n$ corresponding to the span of the column vectors of $A$.

D. Null Space
The null space of an $n \times m$ matrix $A$ is the solution set of the linear system $Ax = 0_{\mathbb{R}^m}$

E. Rank
The rank of an $n \times m$ matrix $A$ is the common dimension of its row and column spaces.

F. Linear Transformation:
A linear transformation is a function $T: \mathbb{R}^m \to \mathbb{R}^n$ such that
(i) $T(\lambda x) = \lambda T(x)$ for all $x \in \mathbb{R}^m$.
(ii) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in \mathbb{R}^m$

G. Range
The range of a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is the following subset of the codomain $\mathbb{R}^n$
$$Range(T) = \{ y \in \mathbb{R}^n \mid y = T(x) \text{ for some } x \in \mathbb{R}^n \}$$

H. Kernel
The kernel of a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is the following subset of the domain $\mathbb{R}^m$
$$Ker(T) = \{ x \in \mathbb{R}^m \mid T(x) = 0_{\mathbb{R}^n} \}$$

II. Using row reduction to identify bases for subspaces

III. Working with Linear Transformations

A. Proving a subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
B. Constructing the $n \times m$ matrix attached to a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$
C. Finding the range and kernel of a linear transformation

IV. Determinants

A. Calculating Determinants using cofactor expansions
B. Calculating Determinants using row reduction
C. Solving square linear systems via Crammer’s Rule
D. Inverting square matrices using cofactors
1. Write down the formal definitions of the following notions:
   
   (a) a \textit{linear transformation from } \mathbb{R}^m \text{ to } \mathbb{R}^n
   
   (b) the \textit{range} of a linear transformation \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \)
   
   (c) the \textit{kernel} of a linear transformation \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \)

2. Consider the following mapping: \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : T ([x_1, x_2, x_3]) = [x_2, x_1 - x_3] \). Show that \( T \) is a linear transformation.

3. Suppose \( T \) is the linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^4 \) given by
   \[ T ([x_1, x_2, x_3]) = [x_1 + x_2, -x_1 + x_3, x_2 + x_3, 0] \]
   
   (a) Find the matrix \( A_T \) such that \( T (x) = Ax \) for all \( x \in \mathbb{R}^3 \).
   
   (b) Find a basis for the \textit{range} of \( T \)
   
   (c) Find a basis for the \textit{kernel} of \( T \).

4. Compute the following determinants by the indicated method
   
   (a) \( \det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) via row reduction
   
   (b) \( \det \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \) via a cofactor expansion

5. Use Crammer’s Rule to solve the following linear system.
   \[
   \begin{align*}
   2x_1 + x_2 &= 5 \\
   x_1 - x_2 &= -2
   \end{align*}
   \]

6. Find the cofactor matrix of the following matrix \( A \) and then use the cofactor matrix to compute \( A^{-1} \).
   \[
   A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}
   \]
1. Write down the formal definitions of the following notions:

(a) a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$

- A linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a function $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that
  \[
  T(x_1 + x_2) = T(x_1) + T(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}^m \]
  \[
  T(\lambda x) = \lambda T(x) \quad \text{for all } \lambda \in \mathbb{R} \text{ and all } x \in \mathbb{R}^m
  \]

(b) the range of a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$

- The range of a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the set
  \[
  \text{Range}(T) = \{ y \in \mathbb{R}^n \mid y = T(x) \text{ for some } x \in \mathbb{R}^m \} \subset \mathbb{R}^n
  \]

(c) the kernel of a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$

- The kernel of a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the set
  \[
  \text{Ker}(T) = \{ x \in \mathbb{R}^m \mid T(x) = 0 \} \subset \mathbb{R}^m
  \]

2. Consider the following mapping: $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_2, x_1 - x_3]$ . Show that $T$ is a linear transformation.

- \[
  T(\lambda [x_1, x_2, x_3]) = T(\lambda x_1, \lambda x_2, \lambda x_3) = [\lambda x_2, \lambda (x_1 - x_3)] = \lambda [x_2, x_1 - x_3] = \lambda T([x_1, x_2, x_3]) \Rightarrow T(\lambda x) = \lambda T(x)
  \]
- \[
  T(x + x') = T([x_1 + x'_1, x_2 + x'_2, x_3 + x'_3]) = [x_2 + x'_2, (x_1 + x'_1) - (x_3 + x'_3)] = [x_2, x_1 - x_3] + [x'_2, x'_1 - x'_3] = T(x) + T(x')
  \]

Since $T$ preserves scalar multiplication and vector addition, $T$ is a linear transformation. \hfill \Box

3. Suppose $T$ is the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by

\[
  T([x_1, x_2, x_3]) = [x_1 + x_2, -x_1 + x_3, x_2 + x_3, 0]
  \]

(a) Find the matrix $A_T$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^3$.

- We first calculate the action of $T$ on the standard basis vectors for the domain $\mathbb{R}^3$:
  \[
  T(e_1) = T([1, 0, 0]) = [1, -1, 0, 0]
  \]
  \[
  T(e_2) = T([0, 1, 0]) = [1, 0, 1, 0]
  \]
  \[
  T(e_3) = T([0, 0, 1]) = [0, 1, 1, 0]
  \]

Converting these to columns gives us the matrix $A_T$

\[
  A_T = \begin{pmatrix}
    1 & 1 & 0 \\
    -1 & 0 & 1 \\
    0 & 1 & 1 \\
    0 & 0 & 0
  \end{pmatrix}
  \]

(b) Find a basis for the range of $T$
• The range of $T$ is equivalent to the column space of $A_T$. To find the latter we first row reduce $A_T$ to reduced row echelon form:

$$
\begin{pmatrix}
1 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\rightarrow 
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

Since we only have pivots in the first two columns of the row echelon form, the first two columns of $A_T$ will provide a basis for the column space of $A_T$, and so also (once reinterpreted as vectors in $\mathbb{R}^4$) a basis for the range of $T$:

basis for \(\text{ColSp}(A_T) = \{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\} \Rightarrow \text{basis for } Range(T) = \{[1, -1, 0, 0], [1, 0, 1, 0]\}\)

(c) Find a basis for the kernel of $T$.

• The kernel of $T$ will correspond to the null space of the matrix $A_T$ (i.e., the solution set of $A_Tx = 0$). Since we have already row reduced $A_T$ to a reduced row echelon form in part (b) above, we can use that RREF for $A_T$ to determine a basis for the null space of $A_T$:

$$
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \text{solution set of }\begin{cases}
x_1 - x_3 = 0 \\
x_2 + x_3 = 0 \\
0 = 0 \\
0 = 0
\end{cases}
$$

Since the third column of the RREF does not contain a pivot, $x_3$ is to be regarded as a free parameter. Writing the general solution vector in terms of the free parameter we get

$$
x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
$$

We can now conclude

basis for \(\text{NullSp}(A_T) = \{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\} \Rightarrow \text{basis for } Ker(T) = \{[1, -1, 1]\}\)

4. Compute the following determinants by the indicated method

(a) \(\det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\) via row reduction

• We have

$$
\begin{align*}
\det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rightarrow R_3 \leftrightarrow R_1 \\
& \rightarrow \det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
& \rightarrow R_3 \rightarrow R_3 - 2R_2 \\
& = \det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
& = - (1)(2)(-3)(1) \\
& = 6
\end{align*}
$$
(the sign flip because we interchanged rows)

(b) \( \det \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \) via a cofactor expansion

- Cofactor expansion along the second row:

\[
\begin{align*}
\det \left( \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \right) &= (0)(-1)^{2+1} \det \left( \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right) + (0)(-1)^{2+1} \det \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) + (2)(-1)^{2+3} \det \left( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\
&= 0 + 0 + (-2)(2) \\
&= 4
\end{align*}
\]

5. Use Crammer’s Rule to solve the following linear system.

\[
\begin{align*}
2x_1 + x_2 &= 5 \\
x_1 - x_2 &= -2
\end{align*}
\]

- Casting this 2 \( \times \) 2 linear system in the form \( A x = b \), we have

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 5 \\ -2 \end{pmatrix}
\]

and

\[
B_1 = \begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} \quad , \quad B_2 = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}
\]

Crammer’s Rule says that the components \( x_1, x_2 \) of the solution vector are given by

\[
x_i = \frac{\det (B_i)}{\det (A)} \quad , \quad i = 1, 2
\]

Now

\[
\begin{align*}
\det (A) &= (2)(-1) - (1)(1) = -3 \\
\det (B_1) &= (5)(-1) - (1)(-2) = -3 \\
\det (B_2) &= (2)(-2) - (5)(1) = -9
\end{align*}
\]

and so

\[
\begin{align*}
x_1 &= \frac{-3}{-3} = 1 \\
x_2 &= \frac{-9}{-3} = 3
\end{align*}
\]

6. Find the cofactor matrix of the following matrix \( A \) and then use the cofactor matrix to compute \( A^{-1} \).

\[
A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}
\]

- We first note that (by a cofactor expansion along the second row of \( A \))

\[
\det (A) = 0 + (1)(-1)^{2+2} \det \left( \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \right) + 0 = 1
\]

The entries of the cofactor matrix of \( A \) are given by

\[
c_{ij} = (-1)^{i+j} \det (A_{ij})
\]
where \( A_{ij} \) is the \( (ij)^{th} \) minor of \( A \). Thus,

\[
\begin{align*}
    c_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = 3, \\
    c_{12} &= (-1)^{1+2} \det \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} = 0, \\
    c_{13} &= (-1)^{1+3} \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \\
    c_{21} &= (-1)^{2+1} \det \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} = 0, \\
    c_{22} &= (-1)^{2+2} \det \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = 1, \\
    c_{23} &= (-1)^{2+3} \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0 \\
    c_{31} &= (-1)^{3+1} \det \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = -2, \\
    c_{32} &= (-1)^{3+2} \det \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 0, \\
    c_{33} &= (-1)^{3+3} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1
\end{align*}
\]

Thus,

\[
C = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}
\]

and

\[
A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{1} \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}
\]