1. (Problems 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5 in text) Find the eigenvalues \( \lambda_i \), the corresponding eigenvectors \( v_i \) of the following matrices. Also find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( D = C^{-1}AC \).

(a) \( A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \)

- First, we calculate the eigenvalues and eigenvectors of \( A \).

\[
0 = \det (A - \lambda I) = \begin{vmatrix} -3 - \lambda & 4 \\ 4 & 3 - \lambda \end{vmatrix} = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5) \Rightarrow \lambda = 5, -5
\]

The eigenspace corresponding to the eigenvalue \( \lambda_1 = 5 \) is the null space of

\[
A - (5)I = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \iff \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]

or, equivalently, the solution space of

\[
2x_1 - x_2 = 0 \quad \Rightarrow \quad x = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)
\]

So the eigenspace corresponding to the eigenvalue \( \lambda_1 = 5 \) is the subspace generated by the vector

\[
v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

The eigenspace corresponding to the eigenvalue \( \lambda_2 = -5 \) is the null space of

\[
A - (-5)I = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \iff \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

or, equivalently, the solution space of

\[
x_1 + 2x_2 = 0 \quad \Rightarrow \quad x = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)
\]

So the eigenspace corresponding to the eigenvalue \( \lambda_2 = -5 \) is the subspace generated by the vector

\[
v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}
\]

Now that we know the eigenvalues and eigenvectors of \( A \), we can write down the diagonal matrix \( D \) by arranging the eigenvalues of \( A \) along the main diagonal of \( D \)

\[
D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}
\]

The matrix \( C \) can be written down by arranging the eigenvectors of \( A \) (in order) as the column vectors of a \( 2 \times 2 \) matrix:

\[
C = [v_1 | v_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}
\]

One can easily verify that

\[
C^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}
\]

and that \( D = C^{-1}AC \) (however, this fact is already guaranteed by the way we constructed the matrices \( D \) and \( C \) ).
(b) \(A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\)

- First, we calculate the eigenvalues and eigenvectors of \(A\).

\[
0 = \det (A - \lambda I) = \begin{vmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) \quad \Rightarrow \quad \lambda = 2, 5
\]

The eigenspace corresponding to the eigenvalue \(\lambda_1 = 2\) is the null space of

\[
A - (2)I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}
\]

or, equivalently, the solution space of

\[
x_1 + 2x_2 = 0 \quad \Rightarrow \quad x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)
\]

So the eigenspace corresponding to the eigenvalue \(\lambda_1 = 2\) is the subspace generated by the vector \(v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\)

The eigenspace corresponding to the eigenvalue \(\lambda_2 = 5\) is the null space of

\[
A - (5)I = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\]

or, equivalently, the solution space of

\[
x_1 - x_2 = 0 \quad \Rightarrow \quad x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]

So the eigenspace corresponding to the eigenvalue \(\lambda_2 = 5\) is the subspace generated by the vector \(v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\)

Now that we know the eigenvalues and eigenvectors of \(A\), we can write down the diagonal matrix \(D\) by arranging the eigenvalues of \(A\) along the main diagonal of \(D\)

\[
D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}
\]

The matrix \(C\) can be written down by arranging the eigenvectors of \(A\) (in order) as the column vectors of a \(2 \times 2\) matrix:

\[
C = [v_1 \mid v_2] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}
\]

(c) \(A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}\)

- First, we calculate the eigenvalues and eigenvectors of \(A\).

\[
0 = \det (A - \lambda I) = \begin{vmatrix} 7 - \lambda & 8 \\ -4 & -5 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \quad \Rightarrow \quad \lambda = 3, -1
\]

The eigenspace corresponding to the eigenvalue \(\lambda_1 = 3\) is the null space of

\[
A - (3)I = \begin{bmatrix} 4 & 8 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}
\]

or, equivalently, the solution space of

\[
x_1 + 2x_2 = 0 \quad \Rightarrow \quad x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)
\]
So the eigenspace corresponding to the eigenvalue $\lambda_1 = 3$ is the subspace generated by the vector

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = -1$ is the null space of

$$A - (-1)I = \begin{bmatrix} 8 & 8 \\ -4 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$x_1 + x_2 = 0$$
$$0 = 0$$

$$\Rightarrow x = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \in \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_2 = -1$ is the subspace generated by the vector

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we know the eigenvalues and eigenvectors of $A$, we can write down the diagonal matrix $D$ by arranging the eigenvalues of $A$ along the main diagonal of $D$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix $C$ can be written down by arranging the eigenvectors of $A$ (in order) as the column vectors of a $2 \times 2$ matrix:

$$C = [v_1 | v_2] = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$

- The characteristic polynomial of $A$ is

$$P_A(\lambda) = \begin{vmatrix} 6 - \lambda & 3 & -3 \\ -2 & -1 - \lambda & 2 \\ 16 & 8 & -7 - \lambda \end{vmatrix} = 3\lambda - 2\lambda^2 - \lambda^3 = -\lambda(\lambda + 3)(\lambda - 1)$$

So $A$ has three distinct real eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -3$ and $\lambda_3 = 1$.

The eigenspace corresponding to the first eigenvector $\lambda_1 = 0$ is the null space of

$$A - (0)I = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$2x_1 + x_2 - x_3 = 0$$
$$x_3 = 0$$
$$0 = 0$$

$$\Rightarrow x_2 \text{ is unfixed}$$
$$x_3 = 0$$

So the corresponding eigenvectors are

$$v_1 \in \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenspace corresponding to the eigenvector $\lambda_2 = -3$ is the null space of

$$A - (2)I = \begin{bmatrix} 9 & 3 & -3 \\ -2 & 2 & 2 \\ 16 & 8 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
is the solution set of
\[\begin{align*}
3x_1 + x_2 - x_3 &= 0 \\
2x_2 + x_3 &= 0 \\
0 &= 0
\end{align*}\]

\[\Rightarrow \begin{align*}
x_1 &= \frac{1}{2}x_3 \\
x_2 &= -\frac{1}{2}x_3 \\
x_3 &\text{ is unfixed}
\end{align*}\]

So the corresponding eigenvectors are

\[v_2 \in \text{span} \left( \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right) \]

The eigenspace corresponding to the first eigenvector \(\lambda_1 = 1\) is the null space of

\[A - (1)I = \begin{bmatrix} 5 & 3 & -3 \\ -2 & -2 & 2 \\ 16 & 8 & -8 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 5 & 3 & -3 \\ 0 & 1 & -1 \end{bmatrix}\]

is the solution set of
\[\begin{align*}
5x_1 + 3x_2 - 3x_3 &= 0 \\
x_2 - x_3 &= 0 \\
0 &= 0
\end{align*}\]

\[\Rightarrow \begin{align*}
x_1 &= 0 \\
x_2 &= x_3 \\
x_3 &\text{ is unfixed}
\end{align*}\]

So the corresponding eigenvectors are

\[v_3 \in \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \]

From the eigenvalues of \(A\) we can now form the diagonal matrix \(D\):

\[D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\]

And from the corresponding eigenvectors we can form the invertible matrix \(C\)

\[C = [v_1 | v_2 | v_3] = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix}\]

such that \(D = C^{-1}AC\).

(e) \(A = \begin{bmatrix} -3 & 10 & -6 \\ 0 & 7 & -6 \\ 0 & 0 & 1 \end{bmatrix}\)

- The characteristic polynomial of \(A\) is

\[P_A(\lambda) = \begin{vmatrix} -3 - \lambda & 10 & -6 \\ 0 & 7 - \lambda & -6 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = -(\lambda + 3)(\lambda - 7)(\lambda - 1)\]

So \(A\) has three distinct real eigenvalues: \(\lambda_1 = -3\), \(\lambda_2 = 7\) and \(\lambda_3 = 1\).

The eigenspace corresponding to the first eigenvector \(\lambda_1 = 0\) is the null space of

\[A - (-3)I = \begin{bmatrix} 0 & 10 & -6 \\ 10 & 0 & -6 \\ 0 & 0 & 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 5 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\]

is the solution set of
\[\begin{align*}
5x_2 - 3x_3 &= 0 \\
x_3 &= 0 \\
0 &= 0
\end{align*}\]

\[x_1 \text{ is unfixed} \]
\[x_2 = 0 \]
\[x_3 = 0 \]
So the corresponding eigenvectors are 

\[ v_1 \in \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \]

The eigenspace corresponding to the eigenvector \( \lambda_2 = 7 \) is the null space of

\[ A - (7)I = \begin{bmatrix} -10 & 10 & -6 \\ 0 & 0 & -6 \\ 0 & 0 & -6 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

is the solution set of

\[
\begin{align*}
\begin{align*}
x_1 - x_2 &= 0 \\
x_3 &= 0 \\
0 &= 0
\end{align*}
\begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 = 0
\end{align*}
\Rightarrow
\begin{align*}
x_2 \text{ is unfixed}
\end{align*}
\]

So the corresponding eigenvectors are 

\[ v_2 \in \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \]

The eigenspace corresponding to the first eigenvector \( \lambda_3 = 1 \) is the null space of

\[ A - (1)I = \begin{bmatrix} -4 & 10 & -6 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \]

is the solution set of

\[
\begin{align*}
\begin{align*}
x_1 - x_3 &= 0 \\
x_2 - x_3 &= 0 \\
0 &= 0
\end{align*}
\begin{align*}
x_1 &= x_3 \\
x_2 &= x_3 = 0
\end{align*}
\Rightarrow
\begin{align*}
x_2 \text{ is unfixed}
\end{align*}
\]

So the corresponding eigenvectors are 

\[ v_3 \in \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \]

From the eigenvalues of \( A \) we can now form the diagonal matrix \( D \):

\[ D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

And from the corresponding eigenvectors we can form the invertible matrix \( C \)

\[ C = [v_1 \mid v_2 \mid v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

such that \( D = C^{-1}AC \).

2. (Problems 5.2.9 and 5.2.10 in text) Determine whether or not the following matrices are diagonalizable.

(a) \( A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix} \)

- Yes, because the matrix is real and symmetric. (See Theorem 5.5 in the text.)

(b) \( A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \)
Let us calculate the characteristic polynomial of $A$:

$$P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 1 & 0 \\ 0 & 3 - \lambda & 1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^3$$

We thus have only one eigenvalue, $\lambda = 3$. The corresponding eigenspace is the null space of

$$A - (3)I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{align*}
x_2 &= 0 \\
x_3 &= 0 \\
x &= \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)
\end{align*}$$

So the eigenspace is just 1-dimensional. But we need three linearly independent eigenvectors to construct the matrix $C$ that diagonalizes $A$. Hence, $A$ is not diagonalizable.

3. (Problem 5.2.13 in text) Mark each of the following True or False.

(a) Every $n \times n$ matrix is diagonalizable.

- False. (An $n \times n$ matrix $A$ needs $n$ linearly independent eigenvectors in order to be diagonalizable.)

(b) If an $n \times n$ matrix has $n$ distinct real eigenvalues, then it is diagonalizable.

- True. (See Theorem 5.3 in text.)

(c) Every $n \times n$ real symmetric matrix is real diagonalizable.

- True. (See Theorem 5.5 in text.)

(d) An $n \times n$ matrix is diagonalizable if and only if it has $n$ real eigenvalues.

- False. (If it has $n$ distinct real eigenvalues then it is diagonalizable, however it is not absolutely necessary that all the eigenvalues are distinct.)

(e) An $n \times n$ matrix is diagonalizable if and only if the algebraic multiplicity of each of its eigenvalues equals the geometric multiplicity.

- True. (See Theorem 5.4 in text.)

(f) Every invertible matrix is diagonalizable.

- False. (Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is invertible since $\det(A) = 1 \neq 0$. However, $\det(A - \lambda I) = (1 - \lambda)^3$ so there is only one eigenvalue. The corresponding eigenspace is the solution space of $(A - I)x = 0$ which is generated by two vectors $[1, 0, 0]$ and $[0, 1, 0]$. However, we need three independent eigenvectors in order to diagonalize $A$. Hence, $A$ is invertible but not diagonalizable.)
(g) Every triangular matrix is diagonalizable.

- False. (See answer to Part (f).)

(h) If $A$ and $B$ are similar matrices and $A$ is diagonalizable, then $B$ is also diagonalizable.

- True.

(i) If an $n \times n$ matrix $A$ is diagonalizable, there is a unique diagonal matrix $D$ that is similar to $A$.

- False. (Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ and $v_1, \ldots, v_n$ are the corresponding set of linearly independent eigenvectors. Then arranging the $\lambda_i$ along the diagonal of an $n \times n$ matrix we obtain a diagonalization $D$ of $A$. However, if we simply changing the ordering of the eigenvalues, then the same procedure produces a different diagonalization of $A$.)

(j) If $A$ and $B$ are similar square matrices then $\det(A) = \det(B)$.

- True. (If $A$ and $B$ are similar, then, by definition, there is an invertible matrix $C$ such that $B = C^{-1}AC$. But then $\det(B) = \det(C^{-1}AC) = \det(C^{-1})\det(A)\det(C) = \det(A)$; since $\det(C^{-1}) = 1/\det(C)$.)