Diagonalization of Matrices

Recall that a **diagonal matrix** is a square \( n \times n \) matrix with non-zero entries only along the diagonal from the under left to the lower right (the *main diagonal*).

Diagonal matrices are particularly convenient for eigenvalue problems since the eigenvalues of a diagonal matrix

\[
A = \begin{bmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & a_{nn}
\end{bmatrix}
\]

coincide with the diagonal entries \( \{a_{ii}\} \) and the eigenvector corresponding the eigenvalue \( a_{ii} \) is just the \( i^{th} \) coordinate vector.

**Example 16.1.** Find the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\]

- The characteristic polynomial is

\[
P_A(\lambda) = \det (A - \lambda I) = \det \begin{bmatrix}
2 - \lambda & 0 \\
0 & 3 - \lambda
\end{bmatrix} = (2 - \lambda)(3 - \lambda)
\]

Evidently \( P_A(\lambda) \) has roots at \( \lambda = 2, 3 \). The eigenvectors corresponding to the eigenvalue \( \lambda = 2 \) are solutions of

\[
(A - (2)I)x = 0 \Rightarrow \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \Rightarrow x_2 = 0
\]

\[
\Rightarrow x \in span \left( \begin{bmatrix}
1 \\
0
\end{bmatrix} \right)
\]

The eigenvectors corresponding to the eigenvalue \( \lambda = 3 \) are solutions of

\[
(A - (3)I)x = 0 \Rightarrow \begin{bmatrix}
-1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \Rightarrow -x_1 = 0
\]

\[
\Rightarrow x \in span \left( \begin{bmatrix}
0 \\
1
\end{bmatrix} \right)
\]

This property (that the eigenvalues of a diagonal matrix coincide with its diagonal entries and the eigenvectors correspond to the corresponding coordinate vectors) is so useful and important that in practice one often tries to make a change of coordinates just so that this will happen. Unfortunately, this is not always possible; however, if it is possible to make a change of coordinates so that a matrix becomes diagonal we say that the matrix is **diagonalizable**. More formally,

**Lemma 16.2.** Let \( A \) be a real (or complex) \( n \times n \) matrix, let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be a set of \( n \) real (respectively, complex) scalars, and let \( v_1, v_2, \ldots, v_n \) be a set of \( n \) vectors in \( \mathbb{R}^n \) (respectively, \( \mathbb{C}^n \)). Let \( C \) be the \( n \times n \)
matrix formed by using $v_j$ for $j^{th}$ column vector, and let $D$ be the $n \times n$ diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$AC = CD$$

if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and each $v_j$ is an eigenvector of $A$ corresponding the eigenvalue $\lambda_j$.

**Proof.** Under the hypotheses

$$AC = A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix}$$

$$CD = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}$$

and so $AC = CD$ implies

$$Av_1 = \lambda_1 v_1$$

$$\vdots$$

$$Av_n = \lambda_n v_n$$

and vice-versa. 

Now suppose $AC = CD$, and the matrix $C$ is invertible. Then we can write

$$D = C^{-1}AC.$$ 

And so we can think of the matrix $C$ as converting $A$ into a diagonal matrix.

**Definition 16.3.** An $n \times n$ matrix $A$ is **diagonalizable** if there is an invertible $n \times n$ matrix $C$ such that $C^{-1}AC$ is a diagonal matrix. The matrix $C$ is said to **diagonalize** $A$.

**Theorem 16.4.** An $n \times n$ matrix $A$ is diagonalizable iff and only if it has $n$ linearly independent eigenvectors.

**Proof.** The argument here is very simple. Suppose $A$ has $n$ linearly independent eigenvectors. Then the matrix $C$ formed by using these eigenvectors as column vectors will be invertible (since the rank of $C$ will be equal to $n$). On the other hand, if $A$ is diagonalizable then, by definition, there must be an invertible matrix $C$ such that $D = C^{-1}AC$ is diagonal. But then the preceding lemma says that the columns vectors of $C$ must coincide with the eigenvectors of $A$. Since $C$ is invertible, these $n$ column vectors must be linearly independent. Hence, $A$ has $n$ linearly independent eigenvectors. 

**Example 16.5.** Find the matrix that diagonalizes

$$A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$$

- First we’ll find the eigenvalues and eigenvectors of $A$.

$$0 = \det (A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 6 \\ 0 & -1 - \lambda \end{bmatrix} = (2 - \lambda)(-1 - \lambda) \Rightarrow \lambda = 2, -1$$

The eigenvectors corresponding to the eigenvalue $\lambda = 2$ are solutions of $(A - (2)I)x = 0$ or

$$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 6x_2 = 0 \Rightarrow x_2 = 0 \Rightarrow x = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenvectors corresponding to the eigenvalue $\lambda = -1$ are solutions of $(A - (-1)I)x = 0$ or

$$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3x_1 + 6x_2 = 0 \Rightarrow x_1 = -2x_2 \Rightarrow x = r \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
So the vectors \( \mathbf{v}_1 = [1, 0] \) and \( \mathbf{v}_2 = [-2, 1] \) will be eigenvectors of \( A \). We now arrange these two vectors as the column vectors of the matrix \( C \).

\[
C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}
\]

In order to compute the diagonalization of \( A \) we also need \( C^{-1} \). This we compute using the technique of Section 1.5:

\[
\begin{bmatrix} 1 & -2 & | & 1 & 0 \\ 0 & 1 & | & 0 & 1 \end{bmatrix} \quad R_1 \rightarrow R_1 + 2R_2 \quad \Rightarrow \quad C^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]

Finally,

\[
D = C^{-1}AC = C^{-1}(AC)
\]

\[
= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}
\]