LECTURE 10

The Rank of a Matrix

Let \( A \) be an \( m \times n \) matrix:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

Recall that the column space of \( A \) is the subspace of \( \mathbb{R}^m \) spanned by the columns of \( A \):

\[
\text{Col}Sp(A) = \text{span} \{ c_1, c_2, \ldots, c_n \} \subset \mathbb{R}^m
\]

where \( i^{th} \) column vector \( c_i \) is defined by

\[
(c_i)_j = a_{ji}, \quad j = 1, \ldots, m
\]

Recall also that the row space of \( A \) is the subspace of \( \mathbb{R}^n \) spanned by the rows of \( A \):

\[
\text{Row}Sp(A) = \text{span} \{ r_1, r_2, \ldots, r_m \} \subset \mathbb{R}^n
\]

where the \( i^{th} \) row vector is defined by

\[
(r_i)_j = a_{ij}, \quad j = 1, \ldots, n
\]

\text{A priori} there is no particular relationship between the column space of \( A \) and the row space of \( A \); indeed, they are not even subspaces of the same space.

**Lemma 10.1.** If a matrix \( A' \) is row equivalent to a matrix \( A \) then the row space of \( A' \) is equal to the row space of \( A \).

**Proof.** First we note that row operations can be built up from row operations of the following form

\[
R_{ij} (\lambda_1, \lambda_2) : \begin{cases}
r_i \rightarrow r_i' = \lambda_1 r_i + \lambda_2 r_j \\
r_i \rightarrow r_i'' = r_i \\
r_i \rightarrow r_i''' = r_i + r_j
\end{cases} \quad i = j, \quad i \neq j, \quad \lambda_1 \neq 0
\]

For example, the interchange of \( i^{th} \) and \( j^{th} \) rows can be carried out as

\[
\begin{bmatrix}
r_i \\
r_j
\end{bmatrix} \xrightarrow{R_{ij}(1,1)} \begin{bmatrix}
r_i' = r_i + r_j \\
r_j' = r_j
\end{bmatrix} \xrightarrow{R_{ij}(-1,-1)} \begin{bmatrix}
r_i'' = r_i' = r_i + r_j \\
r_j'' = -r_j' + r_i' = r_i
\end{bmatrix} \xrightarrow{R_{ij}(1,-1)} \begin{bmatrix}
r_i''' = r_i'' + r_j'' = r_j \\
r_j''' = r_j'' = r_j
\end{bmatrix}
\]

while the other two elementary row operations can be viewed as simply special cases of the row operation (??).

Now suppose \( \mathbf{v} \) is a vector lying in the span of row vectors of \( A \). I will show that it also lies in the span of the row vectors of the matrix \( A' \) obtained by applying the row operation (??) to \( A \).

\[
\mathbf{v} \in \text{span} \{ r_1, r_2, \ldots, r_m \} \Rightarrow \mathbf{v} = c_1 r_1 + \cdots + c_i r_i + \cdots + c_j r_j + \cdots + c_m r_m
\]

\[
= c_1 r_1' + \cdots + c_i \left( \frac{1}{\lambda_1} (r_i' - \lambda_2 r_j) \right) + \cdots + c_j r_j' + \cdots + c_m r_m'
\]

\[
= c_1 r_1' + \cdots + \left( \frac{c_i}{\lambda_1} \right) r_i' + \cdots + \left( c_j - \frac{c_i \lambda_2}{\lambda_1} \right) r_j' + \cdots + c_m r_m'
\]

\[
\in \text{span} \{ r_1', \ldots, r_m' \}
\]

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Thus, the row spaces of A and A' are the same. If A' is row equivalent to A, then by definition there must be a sequence of row operations that converts A into A'.

\[ A \rightarrow A^{(1)} \rightarrow A^{(2)} \rightarrow \cdots \rightarrow A^{(k)} = A' \]

From the preceding paragraph, we know at each intermediate stage we have \( \text{RowSp} \left( A^{(i)} \right) = \text{RowSp} \left( A^{(i+1)} \right) \) so we conclude

\[ \text{RowSp} \left( A' \right) = \text{RowSp} \left( A \right) \]

**Lemma 10.2.** Let A be an \( m \times n \) matrix and let A' be its reduction to row echelon form. Then the non-zero rows of A' form a basis for the row space of A.

The basis idea underlying the proof of this lemma is best illustrated by an example. Suppose A is a 4 \( \times \) 5 matrix that is row equivalent to the following matrix in reduced row-echelon form

\[
A'' = \begin{bmatrix}
1 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Clearly the span of the row vectors of A' is just the span of the first three row vectors (that is to say, the contribution of the last row to the row space of A' is just \( \mathbf{0} \)). On the other hand, it's clear the only way we can satisfy

\[
0 = c_1 \mathbf{r}_1 + c_2 \mathbf{r}_2 + c_3 \mathbf{r}_3
\]

is by taking \( c_1 = c_2 = c_3 = 0 \); because that's the only way to kill off the components of the total sum that come from the pivots of \( \mathbf{r}_1, \mathbf{r}_2 \) and \( \mathbf{r}_3 \) (that is, we can't force a cancellation of terms coming from two different rows because only the pivot row will have a non-zero entry in the component corresponding a column with a pivot). Thus,

\[
0 = c_1 \mathbf{r}_1 + c_2 \mathbf{r}_2 + c_3 \mathbf{r}_3 \quad \Rightarrow \quad c_1 = c_2 = c_3 = 0
\]

\[ \Rightarrow \quad \{ \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \} \text{ is a basis for } \text{span} \left( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \right) = \text{RowSp} \left( A'' \right) = \text{RowSp} \left( A \right) \]

However, this isn't quite the statement of the lemma. For the lemma says the row vectors of a matrix in (un-reduced) echelon form should be a basis for the row space of A. However, we can conclude this simply by noting that

\[
\dim \left( \text{RowSp}(A) \right) = \text{number of vectors in a basis for } \text{RowSp} \left( A \right) \\
= \text{number of non-zero rows in reduced echelon-form } A'' \text{ of } A \\
= \text{number of non-zero rows in an echelon-form } A' \text{ of } A
\]

But because the row vectors of the matrix in echelon-form span \( \text{RowSp} \left( A \right) \), and because the number of these row vectors is the same as the dimension of \( \text{RowSp} \left( A \right) \), we can use Statement 3(b) of Theorem 9.6 (at the end of Lecture 9) to conclude that the row vectors of A' form a basis for \( \text{RowSp} \left( A \right) \).

**Lemma 10.3.** Let A be an \( m \times n \) matrix and let A' be its reduction to row echelon form. Then the columns of A corresponding to the columns of A' containing the pivots of A' form a basis for the column space of A.

This lemma is also best demonstrated by example. Suppose A is a 4 \( \times \) 5 matrix that is row equivalent to the following matrix in reduced row-echelon form

\[
A'' = \begin{bmatrix}
1 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Note that the pivots have been designated by bold-face type. Now the column space of $A''$ will be identical to the row space of its transpose:

$$(A'')^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

where we've used the rows containing the old pivots to clear out the rest of the matrix entries. Obviously, the remaining non-zero rows will be a basis for the $RowSp [(A'')^T] = ColSp |A''| = ColSp |A|$. But the non-zero rows of $(A'')^T$ are just (actually, linear combinations of) of the columns of $A'$ containing pivots. Therefore, the columns of $A'$ that contain pivots correspond to a basis for the column space of $A$.

EXAMPLE 10.4. Find a basis for the column space of

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

- First we row reduce $A$ to row-echelon form:

$$R_1 \leftrightarrow R_2 \\
R_3 \rightarrow R_3 + R_2 \\
R_4 \rightarrow R_4 + R_3$$

The last matrix is a row-echelon form of $A$. It has pivots in the $1^{st}$, $2^{nd}$, and $3^{rd}$ columns. Therefore, the $1^{st}$, $2^{nd}$, and $3^{rd}$ columns of the original matrix $A$ will form a basis for the column space of $A$:

$$ColSp (A) = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

THEOREM 10.5. Let $A$ be an $m \times n$ matrix. The dimension of the row space of $A$ is equal to the dimension of its column space.

This follows easily from the preceding two lemmas since the number of non-zero rows in a matrix in row-echelon form is exactly equal to the number of columns containing pivots. This theorem leads to the following definition.

DEFINITION 10.6. The **rank** of a matrix is the dimension of its row space (which equals the dimension of its column space).

Recall that the null space an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^n$ corresponding to the solution space of $Ax = 0$.

THEOREM 10.7. Let $A$ be an $m \times n$ matrix. Then

$$n = |number \ of \ columns \ of \ A| = \dim |Null \ space \ of \ A| + \text{rank} \ (A)$$

To see why this theorem must be true, consider the following simple example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
This matrix is already in reduced row-echelon form. It has three pivots so
\[\text{rank}(A) = \dim(\text{RowSp}(A)) = \dim(\text{ColSp}(A)) = 3\]
The dimension of its null space is evidently 1 since the solution of the corresponding homogeneous linear system \(Ax = 0\) implies
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\Rightarrow 
\begin{cases}
x_1 = 0 \\
x_2 = 0 \\
x_3 = 0
\end{cases}
\]
but leaves \(x_4\) undetermined. Hence, the dimension of the null space of \(A\) is 1. Thus,
\[4 = \text{number of columns of } A = 3 + 1 = \text{rank of } A \times \dim(\text{null space of } A)\]
In the next lecture we shall develop a geometric interpretation of this fundamental fact.