Math 2233
Homework Set 4

1. Find an integrating factor for each of the following differential equations and obtain the general solution.

(a) \( y + (y - x)y' = 0 \)

- Suppose \( \mu(x, y) \) is an integrating factor for this equation. Then

\[
\mu y + \mu (y - x)y' = 0
\]

must be exact so

\[
\frac{\partial}{\partial y} (\mu y) = \frac{\partial}{\partial x} (\mu y - \mu x)
\]

or

\[
\frac{\partial \mu}{\partial y} y + \mu = \frac{\partial \mu}{\partial x} y - \frac{\partial \mu}{\partial x} x - \mu
\]

In order to simplify this equation we make the hypothesis that \( \mu(x, y) = \mu(y) \). Then \( \frac{\partial \mu}{\partial y} = \frac{d\mu}{dy} \) and \( \frac{\partial \mu}{\partial x} = 0 \); so we have

\[
\frac{d\mu}{dy} y + \mu = -\mu
\]

or

\[
\frac{d\mu}{dy} + \frac{2}{y} \mu = 0.
\]

This is a first order linear equation for \( \mu \). It is also separable, since we can rewrite it as

\[
\frac{d\mu}{\mu} = -\frac{2dy}{y}
\]

Integrating both sides yields

\[
\ln |\mu| = -2 \ln |y|
\]

or

\[
\mu = e^{-2 \ln |y|} = e^{\ln |y^{-2}|} = y^{-2}.
\]

Therefore \( y^{-2} \) should be our integrating factor. Multiplying the original differential equation by \( y^{-2} \) yields

\[
\frac{1}{y} + \left( \frac{1}{y} - \frac{x}{y^2} \right) y' = 0
\]

Now we have

\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left( \frac{1}{y} \right) = -\frac{1}{y^2}
\]

\[
\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{y} - \frac{x}{y^2} \right) = -\frac{1}{y^2}
\]

Since \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \) the equation is exact. Therefore the differential equation is equivalent to a (family of) algebraic equation(s) of the form

\[
\phi(x, y) = C
\]

with

\[
\frac{\partial \phi}{\partial x} = M = \frac{1}{y}
\]

\[
\frac{\partial \phi}{\partial y} = N = \frac{1}{y} - \frac{x}{y^2}
\]
Taking anti-partial derivatives of both these equations yields the following two expressions for \( \phi \):

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int \left( \frac{1}{y} \right) \, dx = \frac{x}{y} + H_1(y)
\]

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int \left( \frac{1}{y} - \frac{x}{y^2} \right) \, dy = \ln |y| + \frac{x}{y} + H_2(x)
\]

Comparing these two expressions we see we must take \( H_1(y) = \ln |y|, H_2(x) = 0 \), and so \( \phi(x, y) = \ln |y| + \frac{x}{y} \). Hence, an implicit solution to the original differential equation is

\[
\ln |y| + \frac{x}{y} = C.
\]

(b) \( x^2 + y^2 + x + yy' = 0 \)

- If \( \mu(x, y) \) is a integrating factor for this equation we must have

\[
\frac{\partial}{\partial y} (\mu x^2 + \mu y^2 + \mu x) = \frac{\partial}{\partial x} (\mu y)
\]

or

\[
\frac{\partial \mu}{\partial y} x^2 + \frac{\partial \mu}{\partial y} y^2 + \mu (2y) + \frac{\partial \mu}{\partial y} x = \frac{\partial \mu}{\partial x} y
\]

If we suppose \( \mu(x, y) \) actually only depends on \( x \) then the above partial differential equation for \( \mu \) simplifies to

\[
2y\mu = y \frac{d\mu}{dx}
\]

or

\[
\frac{d\mu}{\mu} = 2dx
\]

Integrating both sides yields

\[
\ln |\mu| = 2x
\]

or

\[
\mu(x) = e^{2x}.
\]

Hence, \( e^{2x} \) should be an integrating factor for the original differential equation. Hence, 

\[
x^2 e^{2x} + y^2 e^{2x} + x e^{2x} + ye^{2x} y' = 0
\]

should be exact. Indeed,

\[
\frac{\partial}{\partial y} (x^2 e^{2x} + y^2 e^{2x} + xe^{2x}) = 2ye^{2x} = \frac{\partial}{\partial x} (ye^{2x})
\]

so the equation is exact. Therefore we look for a function \( \phi(x, y) \) such that

\[
\frac{\partial \phi}{\partial x} = x^2 e^{2x} + y^2 e^{2x} + xe^{2x}
\]

\[
\frac{\partial \phi}{\partial y} = ye^{2x}
\]

Taking anti-partial derivatives of both these equations yields

\[
\phi(x, y) = \int \left( x^2 e^{2x} + y^2 e^{2x} + xe^{2x} \right) \, dx = \frac{1}{2} x^2 e^{2x} + \frac{1}{2} y^2 e^{2x} + H_1(y)
\]

\[
\phi(x, y) = \int ye^{2x} \, dy = \frac{1}{2} y^2 e^{2x} + H_2(x)
\]

Comparing these two expressions for \( \phi(x, y) \) we see that we must take \( H_1(y) = 0, H_2(x) = \frac{1}{2} x^2 e^{2x} \), and so \( \phi(x, y) = \frac{1}{2} x^2 e^{2x} + \frac{1}{2} y^2 e^{2x} \). Thus, the general (implicit) solution of the original differential equation is

\[
\frac{1}{2} x^2 e^{2x} + \frac{1}{2} y^2 e^{2x} = C
\]
Solving for $y$ yields

$$y(x) = \pm \sqrt{2Ce^{-2x} - x^2}$$

(c) $2y^2 + (2x + 3xy)y' = 0$

- If $\mu(x, y)$ is to be an integrating factor we must have
  $$\frac{\partial}{\partial y} (2y^2 \mu) = \frac{\partial}{\partial x} (2x\mu + 3xy\mu)$$
  or
  $$4y\mu + 2y^2 \frac{\partial \mu}{\partial y} = 2\mu + 2x \frac{\partial \mu}{\partial x} + 3y\mu + 3xy \frac{\partial \mu}{\partial x}$$

This equation would simplify tremendously if we supposed that $\mu(x, y)$ actually depended only on $y$. Then we would have

$$4y\mu + 2y^2 \frac{d\mu}{dy} = 2\mu + 3y\mu$$

or

$$2y^2 \frac{d\mu}{dy} = (2 - y) \mu$$

Integrating both sides yields

$$\ln |\mu| = -\frac{1}{y} - \frac{1}{2} \ln |y|$$

or

$$\mu = \exp \left( -\frac{1}{y} + \ln |y^{-\frac{1}{2}}| \right) = \frac{1}{y^{1/2}} e^{-\frac{x}{y}}.$$

So $\frac{1}{\sqrt{y}} e^{-\frac{1}{y}}$ should be an integrating factor for the original differential equation. Indeed, multiplying the original equation by $\frac{1}{\sqrt{y}} e^{-\frac{1}{y}}$ yields

$$2y^{3/2} e^{-\frac{1}{y}} + \left( 2x \frac{x}{y^{1/2}} e^{-\frac{1}{y}} + 3y^{1/2} x e^{-\frac{1}{y}} \right) y'$$

which is exact since

$$\frac{\partial}{\partial y} \left( 2y^{3/2} e^{-\frac{1}{y}} \right) = 3y^{1/2} e^{-\frac{1}{y}} + 2y^{3/2} \left( \frac{1}{y^2} e^{-\frac{1}{y}} \right) = 3y^{1/2} e^{-\frac{1}{y}} + 2y^{-1/2} e^{-\frac{1}{y}}$$

$$\frac{\partial}{\partial x} \left( 2 \frac{x}{y^{1/2}} e^{-\frac{1}{y}} + 3y^{1/2} x e^{-\frac{1}{y}} \right) = 2y^{-1/2} e^{-\frac{1}{y}} + 3y^{1/2} e^{-\frac{1}{y}}$$

and so we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Now we need to find a $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = 2y^{3/2} e^{-\frac{1}{y}}$$

$$\frac{\partial \phi}{\partial y} = 2x \frac{x}{y^{1/2}} e^{-\frac{1}{y}} + 3y^{1/2} x e^{-\frac{1}{y}}$$

Taking ant-partial derivatives of both of these equations yields

$$\phi(x, y) = 2y^{3/2} e^{-\frac{1}{y}} x + H_1(y)$$

$$\phi(x, y) = 2xy^{3/2} e^{-\frac{1}{y}} + H_2(x)$$

Hence, we must have $H_1(y) = 0$, $H_2(x) = 0$, and $\phi(x, y) = 2y^{3/2} e^{-\frac{1}{y}}$. Thus, the general solution to the original differential equation is given implicitly by

$$2xy^{3/2} e^{-\frac{1}{y}} = C.$$
• If $\mu(x,y)$ is to be an integrating factor we must have
\[
\frac{\partial}{\partial y} (xy\mu) = \frac{\partial}{\partial x} (-x^2 \mu)
\]
or
\[
\mu + xy \frac{\partial \mu}{\partial y} = -2x\mu - x^2 \frac{\partial \mu}{\partial x}.
\]
This equation simplifies tremendously if we can assume $\mu$ depends only on $x$. In this case, the partial differential equation for $\mu$ becomes
\[
x\mu + 0 = -2x\mu - x^2 \frac{d\mu}{dx}
\]
or
\[
x^2 \frac{d\mu}{dx} = -3x\mu
\]
or
\[
\frac{d\mu}{\mu} = -\frac{3}{x} dx
\]
Integrating both sides yields
\[
\ln |\mu| = -3 \ln |x| = \ln |x^{-3}|
\]
or $\mu = x^{-3}$ Multiplying the original differential equation by this function yields
\[
x^{-2}y - x^{-1}y' = 0.
\]
This equation is exact since
\[
\frac{\partial}{\partial y} (x^{-2}y) = x^{-2} \quad \frac{\partial}{\partial x} (-x^{-1}) = x^{-2}
\]
so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Now we look for a $\phi(x, y)$ such that
\[
\frac{\partial \phi}{\partial x} = x^{-2}y \quad \frac{\partial \phi}{\partial y} = -x^{-1}
\]
Taking anti-partial derivatives of each of these equations yields
\[
\phi(x, y) = \int x^{-2}y \, dx = -x^{-1}y + H_1(y)
\]
\[
\phi(x, y) = \int -x^{-1} \, dy = -x^{-1}y + H_2(x).
\]
So we must have $H_1(y) = 0$, $H_2(x) = 0$, and so $\phi(x, y) = x^{-1}y = y/x$. The original differential equation is thus equivalent to an algebraic relation of the form
\[
y = Cx.
\]

2. Solve the following first order differential equations using the substitution $u = y/x$.

(a) $xy' - y = \sqrt{xy}$
This equation is equivalent to

\[ y' = \frac{1}{x} (\sqrt{xy} + y) = \sqrt{\frac{y}{x^2} + \frac{y}{x}} = F \left( \frac{y}{x} \right) \]

where

\[ F(u) \equiv u^{1/2} + u. \]

Thus the equation is homogeneous of degree zero. If we define \( u \equiv y/x \), we have \( u' = y'/x - y/x^2 \) and solving the latter equation for \( y' \) yields

\[ y' = xu' + \frac{y}{x} = xu' + u. \]

If we now substitute \( xu' + u \) for \( y' \) on the left hand side of (1) we obtain

\[ xu' + u = y' = \sqrt{\frac{y}{x} + \left( \frac{u}{x} \right)} = u^{1/2} + u \]

or, cancelling the term \( u \) that appears on both sides,

\[ xu' = u^{1/2} \]

or

\[ u^{-1/2} du = \frac{dx}{x}. \]

Integrating both sides yields

\[ 2u^{1/2} = \ln |x| + C \]

or

\[ 2\sqrt{\frac{y}{x}} = \ln |x| + C. \]

Solving for \( y \) yields

\[ y(x) = \frac{1}{4} x (\ln |x| + C)^2 \]

(b) \( y' = \frac{y^2 + xy}{x^2} \), \( y(1) = 1 \)

This equation is equivalent to

\[ y' = \frac{1}{x^2} \left( \frac{y^2 + xy}{x^2} \right) = \frac{\left( \frac{y}{x} \right)^2 + \left( \frac{y}{x} \right)^2}{1} = \left( \frac{y}{x} \right)^2 + \left( \frac{y}{x} \right) = F \left( \frac{y}{x} \right) \]

where

\[ F(u) \equiv u^2 + u. \]

Substituting \( y = ux \), and \( y' = xu' + u \), we obtain

\[ xu' + u = y' = \left( \frac{y}{x} \right)^2 + \left( \frac{y}{x} \right) = u^2 + u \]

or

\[ xu' = u^2 \]

or

\[ \frac{du}{u^2} = \frac{dx}{x}. \]

Integrating both sides yields

\[ -\frac{1}{u} = \ln |x| + C \]

or

\[ \frac{x}{y} = \ln |x| + C. \]

At this point it is convenient to employ the initial condition \( y(1) = 1 \) to fix the constant of integration \( C \). Setting \( x = 1 \) and \( y = 1 \) in the equation above yields

\[ -1 = \ln |1| + C = 0 + C = C. \]
So $C = 1$. Now we set $C = 1$ in (2) and solve for $y$ to get

$$y(x) = -\frac{x}{\ln |x| - 1}.$$ 

(c) $3xy' + x^2 + y^2 = 0$

- This equation is equivalent to

$$y' = -\frac{x^2 - y^2}{3xy} = -\frac{1}{3} \left( \frac{x}{y} \right) - \frac{1}{3} \left( \frac{y}{x} \right).$$

Making the substitutions $y = ux, y' = xu' + u$ yields

$$xu' + u = -\frac{1}{3} u^{-1} - \frac{1}{3} u = -\frac{1}{3} \left( \frac{1}{u} + u \right) = -\frac{1 - u^2}{3u}$$

or

$$xu' = -\frac{1 - u^2}{3u} - u = -\frac{1 - u^2 - 4u^3}{3u} = -\frac{1}{3} (1 + 4u^2)$$

or, after dividing the extreme sides both by $x (1 + 4u^2),$

$$\frac{u}{1 + 4u^2} \frac{du}{dx} = -\frac{1}{3x}$$

Integrating both sides yields

$$\frac{1}{8} \ln (1 + 4u^2) = \int \frac{udu}{1 + 4u^2} = -\frac{1}{3} \ln |x| + C$$

or

$$\ln \left[ 1 + 4 \left( \frac{y}{x} \right)^2 \right] = C' + \ln |x|^{-8/3}$$

or

$$1 + 4 \left( \frac{y}{x} \right)^2 = \exp \left[ C' + \ln |x|^{-8/3} \right] = e^{C'} \exp \left[ \ln |x|^{-8/3} \right] = C'' x^{-8/3}$$

or, solving for $y,$

$$y(x) = \pm \frac{1}{2x} \sqrt{C'' x^{-8/3} - 1}$$

3. Find a substitution that provides a solution to the following differential equations.

(a) $xy' + y = (xy)^3$

- In the hopes of simplifying the right hand side of the differential equation, we’ll try $z = xy$. In this case, we’ll have $y = z/x$ and $y' = z'/x - z/x^2$. Making the corresponding substitutions into the differential equation yields

$$x \left( \frac{1}{x} \frac{z'}{x^2} - \frac{z}{x^2} \right) + \frac{z}{x} = z^3$$

or

$$z' = z^3$$

or

$$\frac{dz}{z^3} = dx.$$ 

Integrating both sides yields

$$-\frac{1}{2} z^{-2} = x + C$$

or

$$z = \pm \sqrt[3]{C(2x - 2C)}/x.$$
Recalling that $z \equiv xy$, we obtain
\[ yx = \pm \frac{1}{\sqrt{(-2x-2C)}} \]
or
\[ y(x) = \pm \frac{1}{x \sqrt{(-2x-2C)}} \]

(b) $(x + y)y' = (2x + 2y) - 3$

- For this equation it makes sense to try a substitution of the form $z = x + y$. In this case, we would have $y' = z' - 1$ and making the corresponding substitutions into the original differential equation would yield
  \[ z(z' - 1) = 2z - 3 \]
or
  \[ z' = \frac{1}{z} (3z - 3) = \frac{3z - 3}{z} \]
or
  \[ \frac{zdz}{3z - 3} = dx. \]

Integrating both sides yields
\[ = \frac{1}{3} z + \frac{1}{3} \ln (3z - 3) = \int \frac{zdz}{3z - 3} = \int dx + C = x + C \]
or
\[ \frac{1}{3} (x + y) + \frac{1}{3} \ln (3(x + y) - 3) - x = C \]

(Since we won’t be able to solve this equation explicitly for $y$, the best we can do is express the solution as an algebraic equation.)