1. Solve the following initial value problem.

\[ y' - y = 2xe^{2x} ; \quad y(1) = 0 . \]

- This is a first order linear ODE with \( p(x) = -1 \) and \( g(x) = 2xe^{2x} \). So

\[
\mu(x) = \exp \left[ \int p(s) ds \right] = \exp \left[ \int -ds \right] = \exp [-x] = e^{-x}
\]

hence, the general solution of the ODE is

\[
y(x) = \frac{1}{\mu(x)} \int \mu(s) g(s) ds + \frac{C}{\mu(x)}
\]

\[
= \frac{1}{e^{-x}} \int \frac{e^{-s}}{2se^{2s}} ds + \frac{C}{e^{-x}}
\]

\[
= e^x \int 2se^{2s} ds + Ce^x
\]

\[
= e^x (2xe^{2x} - 2e^{2x}) + Ce^x
\]

\[
= 2xe^{2x} - 2e^{2x} + Ce^x
\]

We now impose the initial condition \( y(1) = 0 \):

\[
0 = y(1) = 2(1)e^2 - 2e^2 + Ce^1
\]

\[
= Ce
\]

Thus, \( C = 0 \) and so the solution to the initial value problem is

\[
y(x) = 2xe^{2x} - 2e^{2x}.
\]

2. Solve the following initial value problem.

\[ y' + \frac{2}{x} y = \frac{\cos(x)}{x^2} ; \quad y(\pi) = 0 \]

- This is a first order linear ODE with \( p(x) = \frac{2}{x} \) and \( g(x) = \frac{\cos(x)}{x^2} \). Hence

\[
\mu(x) = \exp \left[ \int p(s) ds \right] = \exp \left[ \int \frac{2}{s} ds \right] = \exp [2 \ln |x|] = \exp [\ln |x^2|] = x^2
\]

and so the general solution of the ODE is

\[
y(x) = \frac{1}{\mu(x)} \int \mu(s) g(s) ds + \frac{C}{\mu(x)}
\]

\[
= \frac{1}{x^2} \int \frac{\cos(s)}{s^2} ds + \frac{C}{x^2}
\]

\[
= \frac{1}{x^2} \cos(s) + \frac{C}{x^2}
\]

\[
= \frac{1}{x^2} \sin(x) + \frac{C}{x^2}
\]

We now impose the initial condition to fix \( C \).

\[
0 = y(\pi) = \frac{1}{\pi^2} \sin(\pi) + \frac{C}{\pi^2} = 0 + \frac{C}{\pi^2} = \frac{C}{\pi^2}
\]

So we must take \( C = 0 \). The solution to the initial value problem is thus

\[
y(x) = \frac{\sin(x)}{x^2}.
\]
3. Find the solution of the initial value problem below. State the interval in which the solution is valid.

\[ xy' + 2y = x^2 - x + 1 \, ; \, y(1) = \frac{1}{2} \, . \]

- Dividing both sides by \( x \) we put the differential equation in standard form:

\[ y' + \frac{2}{x} y = x - 1 + \frac{1}{x} \]

so \( p(x) = \frac{2}{x} \) and \( g(x) = x - 1 + \frac{1}{x} \). Note that since \( p(x) \) and \( g(x) \) are both undefined at \( x = 0 \), we might expect trouble for any solution we construct at the point \( x = 0 \). At any rate

\[ \mu(x) = \exp \left[ \int x p(s) ds \right] = \exp \left[ \int x \frac{2}{s} ds \right] = \exp [2 \ln |x|] = x^2 \]

and so the general solution is

\[ y(x) = \frac{1}{\mu(x)} \int x \mu(s) g(s) ds + \frac{C}{\mu(x)} \]

\[ = \frac{1}{x^2} \int x s^2 \left( s - 1 + \frac{1}{s} \right) ds + \frac{C}{x^2} \]

\[ = \frac{1}{x^2} \int \left( s^3 - s^2 + s \right) ds + \frac{C}{x^2} \]

\[ = \frac{1}{x^2} \left( \frac{1}{4} s^4 - \frac{1}{3} s^3 + \frac{1}{2} s^2 \right) + \frac{C}{x^2} \]

\[ = \frac{1}{4} x^2 - \frac{1}{3} x + \frac{1}{2} + \frac{C}{x^2} \]

Note that if \( C \neq 0 \) then a solution is undefined at \( x = 0 \). Now we plug into the initial condition

\[ \frac{1}{2} = y(1) = \frac{1}{4} (1)^2 - \frac{1}{3} (1) + \frac{1}{2} + \frac{C}{(1)^2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{5}{12} + C \]

so \( C = \frac{1}{12} \). Thus the solution to the initial value problem is

\[ y(x) = \frac{1}{4} x^2 - \frac{1}{3} x + \frac{1}{2} + \frac{1}{12} x^{-2} \]

which is which is well-defined on any interval that excludes the point \( x = 0 \).

4. Find the solution of the initial value problem below. State the interval in which the solution is valid.

\[ y' + y = \frac{1}{1 + x^2} \, , \, y(0) = 0 \, . \]

- The differential equation is in standard form and the coefficient functions \( p(x) = 1 \) and \( g(x) = \frac{1}{1 + x^2} \) are well-defined for all \( x \) so we can expect solutions to be well defined on any subinterval of the real line. Calculating \( \mu(x) \) we get

\[ \mu(x) = \exp \left[ \int x p(s) ds \right] = \exp \left[ \int x ds \right] = e^x \]

and so the general solution will be

\[ y(x) = \frac{1}{\mu(x)} \int x \mu(s) g(s) ds + \frac{C}{\mu(x)} \]

\[ = \frac{1}{e^x} \int x \frac{e^s}{1 + s^2} ds + \frac{C}{e^x} \]

\[ = e^{-x} \int x \frac{e^s}{1 + s^2} ds + \frac{C}{e^x} \]
Unfortunately, there is no way to evaluate the integral
\[ \int \frac{e^x}{1 + x^2} \, dx \]
in closed form. To make further progress, we need to use the following formula for the solution of an initial value problem of the form \( y' + p(x)y = g(x), \) \( y(x_0) = y_o. \)

\[ y(x) = \frac{1}{\mu_o(x)} \int_{x_0}^{x} \mu_o(s)g(s) \, ds + \frac{y_o}{\mu_o(x)} \]

where

\[ \mu_o(x) = \exp \left[ \int_{x_0}^{x} p(s) \, ds \right] \]

Note the use of definite integrals in these formulas. In accordance with the initial condition \( y(0) = 0, \) we set \( x_0 = 0 \) and \( y_o = 0; \) and plug into the formulas (1) and (2):

\[ y(x) = \frac{1}{\mu_o(x)} \int_{x_0}^{x} \mu_o(s)g(s) \, ds + \frac{y_o}{\mu_o(x)} \]

\[ = \frac{1}{e^x} \int_{0}^{x} \frac{e^s}{1 + s^2} \, ds + \frac{0}{e^x} \]

\[ = \frac{1}{e^x} \int_{0}^{x} \frac{e^s}{1 + s^2} \, ds \]

5. Verify that each of the following differential equations is exact and then find the general solution.

(a) \( 2xy \, dx + (x^2 + 1) \, dy = 0 \)

\[ M = 2xy \]
\[ N = x^2 + 1 \]
\[ \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow \text{Exact} \]

Since the differential equation is exact it is equivalent to an algebraic relation of the form

\[ \phi(x, y) = C \]

with

\[ \frac{\partial \phi}{\partial x} = M = 2xy \]
\[ \frac{\partial \phi}{\partial y} = N = x^2 + 1 \]

The most general function \( \phi \) satisfying (3) is obtained taking the anti-partial derivative with respect to \( x; \) i.e., by integrating with respect to \( x, \) treating \( y \) as a constant, and allowing the possibility of an arbitrary function of \( y \) in the result:

\[ \phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int (2xy) \, dx = yx^2 + H_1(y) \]

Similarly, the most general function \( \phi \) satisfying (4) is

\[ \phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int (x^2 + 1) \, dy = x^2y + y + H_2(x) \]
Comparing these two expressions for $\phi(x, y)$ and demanding that they agree with one another, we see that we must take
\[
H_1(y) = y \\
H_2(x) = 0
\]
Hence, $\phi(x, y) = x^2y + y$ and our differential equation is equivalent to the following family of algebraic relations
\[
x^2y + y = C, \text{ with } C \text{ an arbitrary constant}.
\]
Solving this relation for $y$ yields
\[
y(x) = \frac{C}{x^2 + 1}.
\]
(b) $3x^2y \, dx + (x^3 + 1) \, dy = 0$

\[
M = 3x^2y \Rightarrow \frac{\partial M}{\partial y} = 3x^2 \\
N = x^3 + 1 \Rightarrow \frac{\partial N}{\partial x} = 3x^2
\]
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int M(x, y) \, dx = \int 3x^2y \, dx = x^3y + H_1(y) \\
\phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int N(x, y) \, dy = \int (x^3 + 1) \, dy = x^3y + y + H_2(x)
\]
Comparing these two expressions for $\phi(x, y)$ we see that we must take $H_1(y) = y$ and $H_2(x) = 0$.
So $\phi(x, y) = x^3y + y$ and the differential equation is equivalent to
\[
x^3y + y = C
\]
or
\[
y(x) = \frac{C}{1 + x^3}
\]
(c) $y(y + 2x) \, dx + x(2y + x) \, dy = 0$

\[
M = y^2 + 2yx \Rightarrow \frac{\partial M}{\partial y} = 2y + 2x \\
N = 2yx + x^2 \Rightarrow \frac{\partial N}{\partial x} = 2y + 2x
\]
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int M(x, y) \, dx = \int (y^2 + 2yx) \, dx = y^2x + yx^2 + H_1(y) \\
\phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int N(x, y) \, dy = \int (2yx + x^2) \, dy = y^2x + x^2y + H_2(x)
\]
Comparing these two expressions for $\phi(x, y)$ we see that we must take $H_1(y) = 0$ and $H_2(x) = 0$.
So $\phi(x, y) = x^2y + y^2x$ and the differential equation is equivalent to
\[
x^2y + y^2x = C
\]
Solving this equation for $y$ yields
\[
y = \frac{1}{2x} \left( -x^2 \pm \sqrt{x^4 + 4xC^2} \right).
\]
(d) \(y \cos(xy) \, dx + x \cos(xy) \, dy = 0\)

\[
M = y \cos(xy) \Rightarrow \frac{\partial M}{\partial y} = \cos(xy) - xy \sin(xy)
\]
\[
N = x \cos(xy) \Rightarrow \frac{\partial N}{\partial x} = \cos(xy) - xy \sin(xy)
\]

Since \(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\) the equation is exact.

\[
\phi(x, y) = \int_{\partial \phi} \frac{\partial \phi}{\partial x} \, dx = \int M(x, y) \, dx = \int y \cos(xy) \, dx = \sin(xy) + H_1(y)
\]
\[
\phi(x, y) = \int_{\partial \phi} \frac{\partial \phi}{\partial y} \, dy = \int N(x, y) \, dy = \int x \cos(xy) \, dy = \sin(xy) + H_2(x)
\]

Comparing these two expressions for \(\phi(x, y)\) we see that we must take \(H_1(y) = 0, H_2(x) = 0\), and \(\phi(x, y) = \sin(xy)\). Thus the original differential equation is equivalent to

\[
\sin(xy) = C
\]

or

\[
y = \frac{C'}{x}
\]

(Here \(C' = \sin^{-1}(C)\).)

6. Solve the following initial value problems.

(a) \((x - y \cos(x)) - \sin(x)y' = 0\), \(y \left(\frac{\pi}{2}\right) = 1\)

\[
\text{This equation is exact since}
\]
\[
\frac{\partial}{\partial y} (x - y \cos(x)) = -\cos(x) = \frac{\partial}{\partial x} (\sin(x))
\]

Therefore, it must be equivalent to an algebraic equation of the form \(\phi(x, y) = C\) with

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int M(x, y) \, dx = \int (x - y \cos(x)) \, dx = \frac{1}{2} x^2 - y \sin(x) + H_1(y)
\]
\[
\phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int N(x, y) \, dy = \int (\sin(x)) \, dy = y \sin(x) + H_2(x)
\]

Comparing these two expressions for \(\phi(x, y)\) we see we must take \(H_1(y) = 0, H_2(x) = \frac{1}{2} x^2\), and \(\phi(x, y) = \frac{1}{2} x^2 - y \sin(x)\). Hence we must have

\[
\frac{1}{2} x^2 - y \sin(x) = C.
\]

Before solving for \(y\) we’ll impose the initial condition: \(x = \frac{\pi}{2} \Rightarrow y = 1\) to first determine \(C\).

\[
C = \frac{1}{2} x^2 - y \sin(x) = \frac{1}{2} \left(\frac{\pi}{2}\right)^2 - (1) \sin\left(\frac{\pi}{2}\right) = \frac{1}{8} \pi^2 - 1.
\]

Now we solve for \(y\):

\[
y = \frac{\frac{1}{2} x^2 - C}{\sin(x)} = \csc(x) \left(\frac{1}{2} x^2 + 1 - \frac{1}{8} \pi^2\right)
\]

(b) \(x^2 + y^2 + 2xyy' = 0\), \(y(1) = 1\)
• This equation is exact since
\[ \frac{\partial}{\partial y} (x^2 + y^2) = 2y = \frac{\partial}{\partial x} (2xy). \]

Therefore, the differential equation is equivalent to an algebraic relation of the form \( \phi(x, y) = C \) with

\[
\phi(x, y) = \int \frac{\partial \phi}{\partial x} \, dx = \int M(x, y) \, dx = \int \left( x^2 + y^2 \right) \, dx = \frac{1}{3} x^3 - xy^2 + H_1(y) \\
\phi(x, y) = \int \frac{\partial \phi}{\partial y} \, dy = \int N(x, y) \, dy = \int (2xy) \, dy = xy^2 + H_2(x)
\]

Comparing these two expressions for \( \phi(x, y) \) we see we must take \( H_1(y) = 0, \) \( H_2(x) = \frac{1}{3} x^3, \) and so \( \phi(x, y) = \frac{1}{3} x^3 - xy^2. \) We thus have

\[
\frac{1}{3} x^3 + xy^2 = C.
\]

We now impose the initial condition \( x = 1 \Rightarrow y = 1 \) to fix \( C: \)

\[
C = \frac{1}{3} x^3 + xy^2 = \frac{1}{3} (1)^3 + (1)(1)^2 = \frac{4}{3}.
\]

Hence, the differential equation together with the initial condition implies that \( y \) must satisfy

\[
\frac{1}{3} x^3 + xy^2 = \frac{4}{3}.
\]

Solving this equation for \( y \) yields

\[
y = \pm \sqrt{\frac{1}{3x} (4 - x^3)}
\]