LECTURE 17

Homogeneous Equations with Constant Coefficients, Cont’d

Recall that the general solution of a 2\textsuperscript{nd} order linear homogeneous differential equation

\[(17.1) \quad L[y] = y'' + p(x)y' + q(x)y = 0\]

is always a linear combination

\[(17.2) \quad y(x) = c_1 y_1(x) + c_2 y_2(x)\]

of two linearly independent solutions $y_1$ and $y_2$, and we’ve seen that if we’re given one solution $y_1(x)$ we can compute a second linearly independent solution using the method of reduction of order. We will now turn to the problem of actually finding a single solution $y_1(x)$ of (17.1).

We let us now return to the special case of a homogeneous second order linear differential equation with constant coefficients; i.e., differential equations of the form

\[(17.3) \quad y'' + py' + qy = 0\]

where $p$ and $q$ are constant.

We saw in Lecture 11, that one can construct solutions of the differential equation (17.3) by looking for solutions of the form

\[(17.4) \quad y(x) = e^{\lambda x} \ .\]

Let us recall that construction. Plugging (17.4) into (17.3) yields

\[(17.5) \quad 0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = (\lambda^2 + p\lambda + q) \ e^{\lambda x} \ .\]

Since the exponential function $e^{\lambda x}$ never vanishes we must have

\[(17.6) \quad \lambda^2 + p\lambda + q = 0 \ .\]

Equation (17.6) is called the \textbf{characteristic equation} for (17.3) since for any $\lambda$ satisfying (17.6) we will have a solution $y(x) = e^{\lambda x}$ of (17.3).

Now because (17.6) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

\[(17.7) \quad \lambda^2 + p\lambda + q = 0 \quad \Rightarrow \quad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \ .\]

Note that a root $\lambda$ of (17.6) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute $\lambda$ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root $\lambda$ is complex and first discuss the case when the roots of (17.6) are all real. This requires $p^2 - 4q \geq 0$.

Case (i): $p^2 - 4q > 0$
Because \( p^2 - 4q \) is positive, \( \sqrt{p^2 - 4q} \) is a positive real number and

\[
\begin{align*}
\lambda_+ &= \frac{-p + \sqrt{p^2 - 4q}}{2} \\
\lambda_- &= \frac{-p - \sqrt{p^2 - 4q}}{2}
\end{align*}
\]

are distinct real roots of (17.6). Thus,

\[
\begin{align*}
y_1 &= e^{\lambda_+ x} \\
y_2 &= e^{\lambda_- x}
\end{align*}
\]

will both be solutions of (17.3). Noting that

\[
W(y_1, y_2) = \frac{y_1 y'_2 - y'_1 y_2}{\lambda_- e^{\lambda_+ x} e^{\lambda_- x} - \lambda_+ e^{\lambda_+ x} e^{\lambda_- x}}
\]

\[
= \frac{(\lambda_- - \lambda_+) e^{(\lambda_+ + \lambda_-) x}}{\sqrt{p^2 - 4q} e^{-\frac{p}{2} x}}
\]

is non-zero, we conclude that if \( p^2 - 4q \neq 0 \), then the roots (17.8) furnish two linearly independent solutions of (17.3) and so the general solution is given by

\[
y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}
\]

**Case (ii):** \( p^2 - 4q = 0 \)

If \( p^2 - 4q = 0 \), however, this construction only gives us one distinct solution; because in this case \( \lambda_+ = \lambda_- \). To find a second fundamental solution we must use the method of Reduction of Order.

So suppose \( y_1(x) = e^{-\frac{p}{2} x} \) is the solution corresponding to the root

\[
\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm 0}{2} = -\frac{p}{2}
\]

of

\[
\lambda^2 + p\lambda - q = 0, \quad p^2 - 4q = 0.
\]

Then the Reduction of Order formula gives us a second linearly independent solution

\[
y_2(x) = y_1(x) \int^{x}_{\frac{p}{2}} \frac{1}{(y_1(s))^2} \exp \left[ \int^{s}_{\frac{p}{2}} -p(t) dt \right] ds
\]

gives us a second linearly independent solution. Plugging in \( y_1(x) = e^{-\frac{p}{2} x} \) and \( p(t) = p \), yields

\[
y_2(x) = e^{-\frac{p}{2} x} \int^{x}_{\frac{p}{2}} \frac{1}{\left( e^{-\frac{p}{2} s} \right)^2} \exp \left[ \int^{s}_{\frac{p}{2}} -p(t) dt \right] ds
\]

\[
= e^{-\frac{p}{2} x} \int^{x}_{\frac{p}{2}} \frac{1}{e^{-ps}} \exp \left[ -ps \right] ds
\]

\[
= e^{-\frac{p}{2} x} \int^{x}_{\frac{p}{2}} e^{ps} e^{-ps} ds
\]

\[
= e^{-\frac{p}{2} x} \int^{x}_{\frac{p}{2}} ds
\]

\[
= xe^{-\frac{p}{2} x}
\]

\[
= xy_1(x)
\]

In summary, for the case when \( p^2 - 4q = 0 \), we only have one root of the characteristic equation, and so we get only one distinct solution \( y_1(x) \) of the original differential equation by solving the characteristic equation for \( \lambda \). To get a second linearly independent solution we must use the Reduction of Order formula; however, the result will always be the same: **the second linearly independent solution will always be \( x \) times the solution \( y_1(x) = e^{-\frac{p}{2} x} \).** Thus, the general solution in this case will be

\[
y(x) = c_1 e^{-\frac{p}{2} x} + c_2 xe^{-\frac{p}{2} x}, \quad \text{if } p^2 - 4q = 0.
\]
We now turn to the third and last possibility.

Case (iii): \( p^2 - 4q < 0 \)

In this case

\[
(17.12) \quad \sqrt{p^2 - 4q}
\]

will be undefined unless we introduce complex numbers. But when we set

\[
(17.13) \quad i = \sqrt{-1}
\]

we have

\[
(17.14) \quad \sqrt{p^2 - 4q} = \sqrt{-1(4q - p^2)} = i\sqrt{4q - p^2} = i\sqrt{4q - p^2}.
\]

The square root on the right hand side is well-defined since \( 4q - p^2 \) is a positive number. Thus,

\[
(17.15) \quad \lambda_\pm = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta
\]

where

\[
(17.16) \quad \alpha = \frac{b}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2},
\]

will be a complex solution of (17.6) and

\[
(17.17) \quad y(x) = c_1 e^{\alpha x + i\beta x} + c_2 e^{\alpha x - i\beta x}
\]

would be a solution of (17.3) if we could make sense out the notion of an exponential function with a complex argument.

Thus, we must address the problem of assigning some meaning to

\[
(17.18) \quad e^{\alpha x + i\beta x}
\]

as a function of \( x \). To assign some sense to this expression we considered the Taylor series expansion of \( e^x \)

\[
(17.19) \quad e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}x^i.
\]

Now although we do not yet understand what \( e^{\alpha x + i\beta x} \) means, we can nevertheless substitute \( \alpha x + i\beta x \) for \( x \) on the right hand side of (17.19), and get a well defined series with values in the complex numbers. One can show that this series converges for all \( \alpha, \beta \) and \( x \). We thus take

\[
(17.20) \quad e^{\alpha x + i\beta x} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!}(\alpha x + i\beta x)^i
\]

which agrees with (17.19) when \( \beta = 0 \).

One can also show that

\[
(17.21) \quad e^{\alpha x + i\beta x} = e^{\alpha x}e^{i\beta x}.
\]

Thus, when \( p^2 - 4q = 0 \), we have two complex valued solutions to (17.3)

\[
(17.22) \quad y_1(x) = e^{\alpha x}e^{i\beta x} \quad \text{and} \quad y_2(x) = e^{\alpha x}e^{-i\beta x},
\]

where

\[
(17.23) \quad \alpha = \frac{-p}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2}.
\]

A general solution of (17.3) would then be

\[
(17.24) \quad y(x) = c_1 e^{\alpha x}e^{i\beta x} + c_2 e^{\alpha x}e^{i\beta x}.
\]
However, this is rarely the form in which one wants a solution of (17.3). One would prefer solutions that are real-valued functions of $x$ rather than complex-valued functions of $x$. But these can be had as well, since if $z = x + iy$ is a complex number, then

$$
\begin{align*}
\text{Re}(z) &= \frac{1}{2} (z + \bar{z}) = x \\
\text{Im}(z) &= \frac{1}{2i} (z - \bar{z}) = y
\end{align*}
$$

are both real numbers. Applying the Superposition Principle, it is easy to see that if

$$y(x) = e^{\alpha x} e^{i\beta x}
$$

and

$$\bar{y}(x) = e^{\alpha x} e^{-i\beta x}
$$

are two complex-valued solutions of (17.3), then

$$y_r(x) = \frac{1}{2} (y(x) + \bar{y}(x)) = e^{\alpha x} \left( \frac{e^{i\beta x} + e^{-i\beta x}}{2} \right)
$$

and

$$y_i(x) = \frac{1}{2i} (y(x) - \bar{y}(x)) = e^{\alpha x} \left( \frac{e^{i\beta x} - e^{-i\beta x}}{2i} \right)
$$

are both real-valued solutions of (17.3).

Let us now compute the series expansion of

$$
\frac{e^{ix} + e^{-ix}}{2}
$$

and

$$
\frac{e^{ix} - e^{-ix}}{2i}
$$

$$
\begin{align*}
\frac{1}{2} (e^{ix} + e^{-ix}) = & \frac{1}{2} \left( 1 + (-ix) + \frac{1}{2!} (-ix)^2 + \frac{1}{3!} (-ix)^3 + \cdots \right) \\
& \frac{1}{2} \left( 1 + (ix) + \frac{1}{2!} (ix)^2 + \frac{1}{3!} (ix)^3 + \cdots \right) \\
& \frac{1}{2} \left( 1 - \frac{1}{2!} x^2 + \frac{1}{3!} x^4 + \cdots \right)
\end{align*}
$$

The expression on the right hand side is readily identified as the Taylor series expansion of $\cos(x)$. We thus conclude

$$
\cos(x) = \frac{e^{ix} + e^{-ix}}{2}
$$

Similarly, one can show that

$$
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}
$$

On the other hand, if one adds (17.33) to $i$ times (17.34) one gets

$$
\cos(x) + i \sin(x) = \frac{e^{ix} + e^{-ix}}{2} + \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{ix} + e^{-ix}}{2} + \frac{e^{ix} - e^{-ix}}{2} = e^{ix}
$$

or

$$
e^{ix} = \cos(x) + i \sin(x)
$$

Thus, the real part of $e^{ix}$ is $\cos(x)$, while the pure imaginary part of $e^{ix}$ is $\sin(x)$. 
We now have a means of interpreting the function
\[ e^{\alpha x + i\beta x} \]
in terms of elementary functions (rather than as a power series); namely,
\[ e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)) \]
Thus,
\[
\begin{align*}
Re \left[ e^{\alpha x + i\beta x} \right] &= e^{\alpha x} \cos(\beta x), \\
Im \left[ e^{\alpha x + i\beta x} \right] &= e^{\alpha x} \sin(\beta x).
\end{align*}
\]
I now want to show how (17.33) and (17.34) allow us to write down the general solution of a differential equation of the form
\[ y'' + py' + qy = 0 , \quad p^2 - 4q < 0 \]
as a linear combination of real-valued functions.

Now when \( p^2 - 4q < 0 \), then
\[ \lambda = \frac{-p \pm i\sqrt{4q-p^2}}{2} = \alpha \pm i\beta \]
are the (complex) roots of the characteristic equation
\[ \lambda^2 + p\lambda + q = 0 \]
corresponding to (17.40) and
\[ y_{\pm}(x) = e^{\alpha x \pm i\beta} \]
are two (complex-valued) solutions of (17.40). But since (17.40) is linear, since \( y_+ \) and \( y_- \) are solutions so are
\[
\begin{align*}
y_1(x) &= \frac{1}{2} (y_+(x) + y_-(x)) \\
&= \frac{1}{2} \left( e^{\alpha x + i\beta x} + e^{\alpha x - i\beta x} \right) \\
&= e^{\alpha x} \left( \cos(\beta x) \right) \\
&= e^{\alpha x} \cos(\beta x)
\end{align*}
\]
and
\[
\begin{align*}
y_2(x) &= \frac{1}{2} (y_+(x) - y_-(x)) \\
&= \frac{1}{2} \left( e^{\alpha x + i\beta x} - e^{\alpha x - i\beta x} \right) \\
&= e^{\alpha x} \left( \frac{\sin(\beta x)}{2} \right) \\
&= e^{\alpha x} \sin(\beta x)
\end{align*}
\]
Note that \( y_1 \) and \( y_2 \) are both real-valued functions.

We conclude that if the characteristic equation corresponding to
\[ y'' + py' + qy = 0 \]
has two complex roots
\[ \lambda = \alpha \pm i\beta \]
then the general solution is
\[ y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) \]
Example 17.1. The differential equation
\[ y'' - 2y' - 3y \]
has as its characteristic equation
\[ \lambda^2 - 2\lambda - 3 = 0 \]
The roots of the characteristic equation are given by
\[ \lambda = \frac{2 \pm \sqrt{4 + 12}}{2} = 3, -1 \]
These are distinct real roots, so the general solution is
\[ y(x) = c_1 e^{3x} + c_2 e^{-x} \]

Example 17.2. The differential equation
\[ y'' + 4y' + 4y = 0 \]
has
\[ \lambda^2 + 4\lambda + 4 = 0 \]
as its characteristic equation. The roots of the characteristic equation are given by
\[ \lambda = \frac{-4 \pm \sqrt{16 - 16}}{2} = -2 \]
Thus we have a double root and the general solution is
\[ y(x) = c_1 e^{-2x} + c_2 xe^{-2x} \]

Example 17.3. The differential equation
\[ y'' + y' + y = 0 \]
has
\[ \lambda^2 + \lambda + 1 = 0 \]
as its characteristic equation. The roots of the characteristic equation are
\[ \lambda = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \]
and so the general solution is
\[ y(x) = c_1 e^{-\frac{1}{2}x} \cos \left( \frac{\sqrt{3}}{2} x \right) + c_2 e^{-\frac{1}{2}x} \sin \left( \frac{\sqrt{3}}{2} x \right) \]