LECTURE 14

Second Order Linear Equations, General Theory

1. Standard Form

A second order linear differential equation is a differential equation of the form

\[ A(x)y'' + B(x)y' + C(x)y = D(x) \]  

(Here \( A, B, C \) and \( D \) are certain prescribed functions of \( x \).)

As in the case of first order linear equations, in any interval where \( A(x) \neq 0 \), we can replace such an equation by an equivalent one in standard form:

\[ y'' + p(x)y' + q(x)y = g(x) \]  

where

\[ p(x) = \frac{B(x)}{A(x)} \]

\[ q(x) = \frac{C(x)}{A(x)} \]

\[ g(x) = \frac{D(x)}{A(x)} \]

2. Homogeneous vs. Non-homogeneous Linear Differential Equations

In the development that follows it will be important to distinguish between the case when the right hand side of

\[ y'' + p(x)y' + q(x)y = g(x) \]  

is zero or non-zero. We shall say that a second order linear ODE is homogeneous if it can be written in the form

\[ y'' + p(x)y' + q(x)y = 0 \]  

otherwise (if \( g(x) \neq 0 \)) we shall say that it is non-homogeneous. Note that this terminology is completely unrelated to homogeneous equations of degree zero (the topic of the preceding lecture).

3. Differential Operator Notation

Consider the general second order linear differential equation

\[ \phi'' + p(x)\phi' + q(x)\phi = g(x) \]  

We shall often write differential equations like this as

\[ L[\phi] = g(x) \]  

where \( L \) is the linear differential operator

\[ L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \]
4. General Theorems

That is to say, \( L \) is the operator that acts on a function \( \phi \) by

\[
L[\phi] = \left( \frac{d^2 \phi}{dx^2} + p(x) \frac{d \phi}{dx} + q(x) \right) \phi = \frac{d^2 \phi}{dx^2} + p(x) \frac{d \phi}{dx} + q(x) \phi .
\]

The following theorem tells us the conditions for the existence and uniqueness of solutions of a second order linear differential equation.

**Theorem 14.1.** If the functions \( p, q \) and \( g \) are continuous on an open interval \( I \subset \mathbb{R} \) containing the point \( x_o \), then in some interval about \( x_o \) there exists a unique solution \( y = \phi(x) \) to the differential equation

\[
y'' + p(x)y' + q(x)y = g(x)
\]

satisfying the prescribed initial conditions

\[
y(x_o) = y_o, \quad y'(x_o) = y'_o .
\]

Note how this theorem is analogous to the corresponding theorem for first order linear ODE’s. Note also that the conditions for existence and uniqueness are fairly lax - all we require is the continuity of the functions \( p, q, \) and \( g \) around a given initial point. Finally, we note that the form of the initial conditions involves the specification of both \( y(x) \) and its derivative \( y'(x) \) at an initial point \( x_o \).

I should also point out that the preceding theorem does not address the issue of how to construct a solution of a second order linear ODE. Indeed, the actual construction of solutions to second order linear ODE is sufficiently complicated to that we shall spend 90% of the remaining lectures on techniques of solution. The next two theorems at least tell us the basic ingredients for a general solution of a second order linear ODE.

**Theorem 14.2.** (The Superposition Principle) If \( y = y_1(x) \) and \( y = y_2(x) \) are two solutions of the differential equation

\[
L[y] = \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0
\]

then any linear combination

\[
y = c_1 y_1(x) + c_2 y_2(x)
\]

of \( y_1(x) \) and \( y_2(x) \), where \( c_1 \) and \( c_2 \) are constants, is also a solution of (14.12).

**Proof.**

\[
L[c_1 y_1 + c_2 y_2] = \frac{d^2}{dx^2} (c_1 y_1 + c_2 y_2) + p(x) \frac{d}{dx} (c_1 y_1 + c_2 y_2) + q(x) (c_1 y_1 + c_2 y_2)
\]

\[
= c_1 \left( \frac{d^2 y_1}{dx^2} + p(x) \frac{dy_1}{dx} + q(x)y_1 \right) + c_2 \left( \frac{d^2 y_2}{dx^2} + p(x) \frac{dy_2}{dx} + q(x)y_2 \right)
\]

\[
= c_1 \cdot 0 + c_2 \cdot 0 = 0
\]

The fact that a linear combination of solutions of a **linear, homogeneous differential equation** is also a solution is extremely important. The theory of linear homogeneous equations, including differential equations involving higher derivatives depends strongly on the superposition principle.
Example 14.3.

\begin{equation}
 y_1(x) = \cos(x)
\end{equation}

and

\begin{equation}
 y_2(x) = \sin(x)
\end{equation}

are both solutions of

\begin{equation}
 y'' + y = 0.
\end{equation}

It is easy to check that any linear combination of $y_1$ and $y_2$ is also a solution.

Example 14.4.

\begin{equation}
 y_1(x) = 1
\end{equation}

and

\begin{equation}
 y_2(x) = x^{1/2}
\end{equation}

are both solutions of

\begin{equation}
 yy'' + (y')^2 = 0.
\end{equation}

However, it is easy to check that $y_1 + y_2 = 1 + \sqrt{x}$ is not a solution of (14.20). The reason for this lies in the fact that (14.20) is not linear.

Given two solutions $y_1$ and $y_2$ of a second order linear homogeneous differential equation

\begin{equation}
 L[y] = 0,
\end{equation}

we can construct an infinite number of other solutions

\begin{equation}
 y(x) = c_1 y_1(x) + c_2 y_2(x)
\end{equation}

by letting $c_1$ and $c_2$ run through $\mathbb{R}$. The following question then arises: are all the solutions of (14.21) capable of being expressed in form (14.22) for some choice of $c_1$ and $c_2$?

This will not always be the case; and so we shall say that two solutions $y_1$ and $y_2$ form a fundamental set of solutions to (14.21) if every solution of (14.21) can be expressed as a linear combination of $y_1$ and $y_2$.

**Theorem 14.5.** If $p$ and $q$ are continuous on an open interval $I = (\alpha, \beta)$ and if $y_1$ and $y_2$ are solutions of the differential equation

\begin{equation}
 L[y] = y'' + p(x)y' + q(x)y = 0
\end{equation}

satisfying

\begin{equation}
 W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0
\end{equation}

at every point $x \in I$, then any other solution of (14.23) on the interval $I$ can be expressed uniquely as a linear combination of $y_1$ and $y_2$.

**Proof.**

Let $y_1$ and $y_2$ be two given solutions on an interval $I$ and let $Y$ be any other solution on $I$. Choose a point $x_o \in I$. From our basic uniqueness and existence theorem (Theorem 3.2), we know that there is only solution $y(x)$ of (14.23) such that

\begin{align*}
 y(x_o) &= Y'(x_o) \\
 y'(x_o) &= Y'(x_o)
\end{align*}

namely, $Y(x)$. Therefore if we can show that a solution of the form

$$
y(x) = c_1 y_1(x) + c_2 y_2(x)
$$
satisfies the initial conditions (14.25), then we must have $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ and so $Y(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$. 
Thus, we now seek to define constants $c_1$ and $c_2$ so that these initial conditions can be matched. We thus set

\begin{equation}
(14.26) \quad c_1 y_1(x_o) + c_2 y_2(x_o) = y_o \\
c_1 y'_1(x_o) + c_2 y'_2(x_o) = y'_o .
\end{equation}

This is just a series of two equations with two unknowns. Solving the first equation for $c_1$ yields

\begin{equation}
(14.27) \quad c_1 = \frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} .
\end{equation}

Plugging this into the second equation yields

\begin{equation}
(14.28) \quad \frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} y'_1(x_o) + c_2 y'_2(x_o) = y'_o
\end{equation}
or

\begin{equation}
(14.29) \quad y_o y'_1(x_o) - c_2 y_2(x_o) y'_1(x_o) + c_2 y_1(x_o) y'_2(x_o) = y_1(x_o) y'_o
\end{equation}
or

\begin{equation}
(14.30) \quad c_2 = \frac{y_1(x_o) y'_o - y'_1(x_o) y_o}{y_1(x_o) y'_2(x_o) - y'_1(x_o) y_2(x_o)} .
\end{equation}

Plugging this expression for $c_2$ into (14.27) yields

\begin{equation}
(14.31) \quad c_1 = \frac{y_o y'_2(x_o) - y_2(x_o) y'_o}{y_1(x_o) y'_2(x_o) - y'_1(x_o) y_2(x_o)} .
\end{equation}

Thus, we can solve for $c_1$ and $c_2$ whenever the denominator

\begin{equation}
(14.32) \quad W(y_1, y_2) = y_1(x_o) y'_2(x_o) - y'_1(x_o) y_2(x_o)
\end{equation}
does not vanish. Thus, so long as $y_1$ and $y_2$ satisfy (14.23) we can always express any solution as a linear combination of $y_1$ and $y_2$.  

Remark: The quantity

\begin{equation}
(14.33) \quad W(y_1, y_2) = y_1(x_o) y'_2(x_o) - y'_1(x_o) y_2(x_o)
\end{equation}
is called the Wronskian of $y_1$ and $y_2$.

Example 14.6. Show that

\begin{equation}
(14.34) \quad y_1(x) = \cos(x)
\end{equation}
and

\begin{equation}
(14.35) \quad y_2(x) = \sin(x)
\end{equation}
are form a set of fundamental solutions to the differential equation

\begin{equation}
(14.36) \quad y'' + y = 0 .
\end{equation}

We simply have to check that the Wronskian does not vanish:

\begin{align}
W(y_1, y_2) &= y_1(x_o) y'_2(x_o) - y'_1(x_o) y_2(x_o) \\
&= \cos(x) \cos(x) - (- \sin(x)) \sin(x) \\
&= 1 \\
&\neq 0 .
\end{align}

Since the Wronskian does not vanish, $y_1$ and $y_2$ must be linearly independent.