Math 2233

Solutions to Quiz 3

1. Reduce the following expressions to a single power series expression.

(a) $\sum_{n=0}^{\infty} n(n-1)a_n x^n - x\sum_{n=0}^{\infty} a_n x^n$

- $$\sum_{n=0}^{\infty} n(n-1)a_n x^n - x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} -a_n x^{n+1}$$

  after moving the factor of x through the second summation;

  $$= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n$$

  after shifting the second summation index by $n \to n-1$;

  $$= 0 + \sum_{n=1}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} -a_{n-1}x^n$$

  after peeling off the first $(n=0)$ term of the first summation;

  $$= \sum_{n=1}^{\infty} \left[(n(n-1))a_n - a_{n-1}\right]x^n$$

  after adding the two power series.

(b) $x\sum_{n=1}^{\infty} na_n (x-1)^n$

- Here the problem is that a power series is being multiplied by a function of x. To handle this we replace x by its Taylor expansion about $x=1$; because the power series it's multiplying is a power seires about $x=1$. If $f(x)=x$

  $$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\ &= 1 + (x-1) \end{aligned}$$

  because $f(1)=1$, $f'(1)=1$ and all higher derivatives are 0 (for $f(x)=x$). Thus,

  $$\begin{aligned} x\sum_{n=1}^{\infty} na_n(x-1)^n &= (1+(x-1))\sum_{n=1}^{\infty} na_n(x-1)^n \\ &= \sum_{n=1}^{\infty} na_n(x-1)^n + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} \end{aligned}$$

  Now we have to add these two power series about $x=1$. So first we do a shift of summation index, $n \to n-1$, on the second series

  $$= \sum_{n=1}^{\infty} na_n(x-1)^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n$$

  Then we peel of the first term of the first series (since it begins before the first term of the second series)

  $$\begin{aligned} &= (1)a_1(x-1) + \sum_{n=2}^{\infty} na_n(x-1)^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n \\ &= a_1(x-1) + \sum_{n=2}^{\infty}\left[na_n + (n-1)a_{n-1}\right](x-1)^n \end{aligned}$$

2. Consider the differential equation

(*) $$y'' - 2xy' + y = 0$$

(a) Find the Recursion Relations for a power series solution of (*) expanded about $x_0 = 0$.

- We set

$$\begin{aligned} y &= \sum_{n=0} a_n x^n \\ \Rightarrow &\quad -2xy' = -2x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (-2) n a_n x^n \\ \Rightarrow &\quad y'' = \sum_{n=0}^{\infty} n(n-1) x^{n-2} = 0 + 0 + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \end{aligned}$$

To the differential equation requires

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (-2) n a_n x^n + \sum_{n=0} a_n x^n$$

or

$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n + a_n\right] x^n$$

$$\Rightarrow \quad (n+2)(n+1) a_{n+2} - 2n a_n + a_n = 0 \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Or

$$\Rightarrow \quad a_{n+2} = \frac{(2n-1)}{(n+2)(n+1)} a_n \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

(The last equations are the recursion relations).

(b) Write down the Taylor series for the solution of (*) satisfying $y(0) = 1$, $y'(0) = 0$ up to terms of order x^4.

- The initial conditions at $x = 0$ provide the first two coefficients of the power series solution about $x = 0$:

$$\begin{aligned} y(0) &= 1 \quad \Rightarrow \quad a_0 = 1 \\ y'(0) &= 0 \quad \Rightarrow \quad a_1 = 0 \end{aligned}$$

We can now build up the higher coefficients using the recursion relations

$$\begin{aligned} a_2 &= a_{0+2} = \frac{2(0) - 1}{(0+2)(0+1)} a_0 = -\frac{1}{2} a_0 = -\frac{1}{2} \\ a_3 &= a_{1+2} = \frac{2(1) - 1}{(1+2)(1+1)} a_1 = \frac{1}{6} a_1 = 0 \\ a_4 &= a_{2+2} = \frac{2(2) - 1}{(2+2)(2+1)} a_2 = \frac{3}{12} a_2 = \frac{3}{12}\left(-\frac{1}{2}\right) = -\frac{1}{8} \end{aligned}$$

Thus, our solution to order x^4 is

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^4 + a_4 x^4 + \cdots \\ &= 1 + 0x - \frac{1}{2} x^2 + 0x^3 - \frac{1}{8} x^4 + \cdots \\ &= 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 + \cdots \end{aligned}$$