Math 2233

SOLUTIONS TO SECOND EXAM

9:00 - 10:15 am, July 10, 2014

1. Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^2$ are solutions to $x^2y'' - 2y = 0$.

(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

- $$W[y_1, y_2] \equiv y_1y_2' - y_1'y_2 = \left(x^{-1}\right)(2x) - \left(-x^{-2}\right)\left(x^2\right) = 2 - (-1) = 3 \neq 0$$
  Since their Wronskian doesn't vanish, y_1 and y_2 are linearly independent functions. □

(b) (5 pts) Write down the general solution of $x^2y'' - 2y = 0$.

- $$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^{-1} + c_2x^2$$
  □

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 3$, $y'(1) = 1$.

- $$\left.\begin{array}{c} 3 = y(1) = c_1 + c_2 \\ 1 = y'(1) = \left(-c_1x^{-2} + 2c_2x\right)\big|_{x=0} = -c_1 + 2c_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c_1 + c_2 = 3 \\ -c_1 + 2c_2 = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 = \frac{5}{3} \\ c_2 = \frac{4}{3} \end{array}\right.$$
  $$\Rightarrow \quad y(x) = \frac{5}{3}x^{-1} + \frac{4}{3}x^2$$
  □

2. (15 pts) Given that $y_1(x) = x^3$ is one solution of $x^2y'' - 5xy' + 9y = 0$, use Reduction of Order(explicitly) to determine the general solution.

- $$\begin{aligned} y_2 &= y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int^x p\,ds\right) dx \\ &= x^3 \int \frac{1}{(x^3)^2} \exp\left(+\int^x \frac{5}{s} dx\right) \\ &= x^3 \int x^{-6} \exp\left(5\ln|x|\right) dx \\ &= x^3 \int \left(x^{-6}\right) x^5 dx = x^3 \int \frac{1}{x} dx = x^3 \ln|x| \\ \Rightarrow \quad & y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^3 + c_2x^3\ln|x| \end{aligned}$$
  □

3. Given that $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are solutions of $y'' + y' - 2y = 0$.

(a) (10 pts) Use the Method of Variation of Parameters to find the general solution of

$$y'' + y' - 2y = e^{-x} \quad .$$

- We have

$$g(x) = e^{-x} \text{ and } W[y_1, y_2] = (e^x)\left(-2e^{-2x}\right) - (e^x)\left(e^{-2x}\right) = -3e^{-x}$$

$$y_p(x) = -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx = -e^x \int \frac{\left(e^{-2x}\right) e^{-x}}{-3e^{-x}} dx + e^{-2x} \int \frac{(e^x)\left(e^{-x}\right)}{-3e^{-x}} dx$$

$$= \frac{1}{3} e^x \int e^{-2x} dx - \frac{1}{3} e^{-2x} \int e^x dx = \frac{1}{3} e^x \left(-\frac{1}{2} e^{-2x}\right) - \frac{1}{3} e^{-2x} (e^x) = -\frac{1}{2} e^{-x}$$

$$\Rightarrow \quad y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = -\frac{1}{2} e^{-x} + c_1 e^x + c_2 e^{-2x}$$

□

(b) (5 pts) Find the solution satisfying $y(0) = 1$, $y'(0) = 0$.

- If $y = -\frac{1}{2} e^{-x} + c_1 e^x + c_2 e^{-2x}$, then $y' = \frac{1}{2} e^{-x} + c_1 e^x - 2c_2 e^{-2x}$ and so

$$\left.\begin{array}{l} 1 = y(0) = -\frac{1}{2} + c_1 + c_2 \\ 0 = y'(0) = \frac{1}{2} + c_1 - 2c_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c_1 + c_2 = \frac{3}{2} \\ c_1 - 2c_2 = -\frac{1}{2} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 = \frac{5}{6} \\ c_2 = \frac{2}{3} \end{array}\right.$$

$$y(x) = -\frac{1}{2} e^{-x} + \frac{5}{6} e^x + \frac{2}{3} e^{-2x}$$

□

4. (15 pts) Suppose you know that a particular function $y_1(x)$ is a solution of $y'' + p(x)y' + q(x)y = 0$, explain the computational steps by which one can construct a formula for the general solution of $y'' + p(x)y' + q(x)y = (x)$. (You need not carry out any explicit computations but **do** write down the relevant formulas.)

- (i) One first calculates a second independent solution of the homogeneous equation using Reduction of Order:

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[-\int^s p(t)\, dt\right] ds$$

(ii) Next, one uses the two independent solutions of the homogeneous equation to calculate a particular solution of the nonhomogeneous equation $(g(x) \neq 0)$ via the Variation of Parameters formula:

$$y_p(x) = -y_1(x) \int^x \frac{y_2(s)\, g(s)}{y_1(s)\, y_2'(s) - y_1'(s)\, y_2(s)} ds + y_2(x) \int^x \frac{y_2(s)\, g(s)}{y_1(s)\, y_2'(s) - y_1'(s)\, y_2(s)} ds$$

(iii) Finally, with y_1, y_2, and y_p in hand, one can write down the general solution of the nonhomogeneous equation as

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$

5. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' + 2y' + 5y = 0$

- $$0 = \lambda^2 - 2\lambda + 5 \quad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$
$$\Rightarrow \quad y(x) = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)$$

□

(b) (5 pts) $y'' - 5y' + y = 0$

- $$0 = \lambda^2 - 5\lambda + y \quad \Rightarrow \quad \lambda = \frac{5 \pm \sqrt{25-4}}{2} = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$$
$$\Rightarrow \quad y(x) = c_1 e^{\left(\frac{5+\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{5-\sqrt{21}}{2}\right)x}$$

□

(c) (5 pts) $4y'' - 4y' + y = 0$

- $$0 = 4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 \quad \Rightarrow \quad \lambda = \frac{1}{2}$$
$$\Rightarrow \quad y(x) = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$

□

6. Find the general solution of the following Euler-type differential equations.

(a) (5 pts) $x^2y'' - 4xy - 6y = 0$

- $$0 = r(r-1) - 4r - 6 = r^2 - 5r - 6 = (r-6)(r+1) \quad \Rightarrow \quad r = 6, -1$$
$$\Rightarrow \quad y(x) = c_1 x^6 + c_2 x^{-1}$$

□

(b) (5 pts) $x^2y'' + 11xy' + 25y = 0$

- $$0 = r(r-1) + 11r + 25 = r^2 + 10r + 25 = (r+5)^2 \quad \Rightarrow \quad r = -5$$
$$\Rightarrow \quad y(x) = c_1 x^{-5} + c_2 x^{-5}\ln|x|$$

□

(c) (5 pts) $x^2y'' - xy' + 3y = 0$

- $$0 = r(r-1) - r + 3 = r^2 - 2r + 3 \quad \Rightarrow \quad r = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \sqrt{2}i$$
$$\Rightarrow \quad y(x) = c_1 x\cos\left(\sqrt{2}\ln|x|\right) + c_2 x\sin\left(\sqrt{2}\ln|x|\right)$$

□