Math 2233
SOLUTIONS TO FIRST EXAM
June 30, 2014

1. Classify the following differential equations by determining their order, determining whether they are linear or non-linear differential equations, and determining if they are ordinary differential equations or partial differential equations.

(a) (5 pts) $y'' + x^2 y = e^{yx}$

- 2^{nd} order, nonlinear, ODE

(b) (5 pts) $\frac{\partial^2 \phi}{\partial x^2} + y^2 \frac{\partial \phi}{\partial y} = \phi$

- 2^{nd} order, linear, PDE

(c) (5 pts) $s\frac{d^3 s}{dt^3} + t^2 \frac{ds}{dt} + s = t$

- 3^{rd} order, non-linear, ODE

2.
(a) (5 pts) Give an example of a **separable** first order ODE.

- $M(x) = N(x) \frac{dy}{dx}$.

(b) (5 pts) Give an example of a **linear** first order ODE.

- $\frac{dy}{dx} + p(x)\, y = g(x)$

(c) (5 pts) Given an example of an **exact** first order ODE and demonstrate that it is exact.

- $M(x,y) + N(x,y) \frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (e.g. $x + \sin(y) + (x\cos(y) + y) \frac{dy}{dx} = 0$)

3. (15 pts) Find an explicit solution of the following (separable) differential equation.

$$2x - e^{2y} y' = 0$$

- We have

$$\begin{aligned} e^{2y}\frac{dy}{dx} &= 2x \\ &\Rightarrow \int e^{2y} dy = \int 2x \; dx + C \\ &\Rightarrow \frac{1}{2} e^{2y} = x^2 + C \\ &\Rightarrow e^{2y} = 2x^2 + 2C \\ &\Rightarrow y = \frac{1}{2}\ln\left(2x^2 + C'\right) \end{aligned}$$

4. (15 pts) Solve the following initial value problem

$$xy' - 3y = x^2 \qquad , \qquad y(1) = 2$$

- This differential equation is first order linear equivalent to the following 1^{st} order linear equation in standard form

$$\frac{dy}{dx} - \frac{3}{x}y' = x \quad \Rightarrow \quad p(x) = -\frac{3}{x} \quad , \quad g(x) = x \quad .$$

We first calculate the integrating factor $\mu(x)$:

$$\mu(x) = \exp\left(\int p(x)\,dx\right) = \exp\left(\int\left(-\frac{3}{x}\right)dx\right) = \exp\left(-3\ln|x|\right) = x^{-3}$$

Next we calculate the general solution

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)} = \frac{1}{x^{-3}}\int x^{-3}(x)\,dx + \frac{C}{x^{-3}} \\ &= x^3\int x^{-2}dx + Cx^3 = x^3\left(-\frac{1}{x}\right) + Cx^3 \\ &= -x^2 + Cx^3 \end{aligned}$$

Finally, we impose the boundary condition to fix the choice of C :

$$2 = y(1) = -(1)^2 + C(1)^3 = -1 + C \quad \Rightarrow \quad C = 3$$

The unique solution is thus

$$y(x) = -x^2 + 3x^3$$

5. (15 pts) Find an implicit solution to the following initial value problem (Hint: the differential equation is exact.)

$$\frac{y}{x} + 2x + \ln|x|\frac{dy}{dx} = 0 \qquad , \qquad y(2) = 1$$

- First, we verify that equation is exact

$$\begin{aligned} M(x,y) &= \frac{y}{x} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{1}{x} \\ N(x,y) &= \ln|x| \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{1}{x} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact. This means that its solutions are the same as the solutions of an algebraic equation $\phi(x,y) = C$ with $\phi(x,y)$ determined by

$$\begin{aligned} \phi(x,y) &= \int M(x,y)\,\partial x + H_1(y) = \int\left(\frac{y}{x} + 2x\right)\partial x + H_1(y) = y\ln|x| + x^2 + H_1(y) \\ \phi(x,y) &= \int N(x,y)\,\partial y + H_2(x) = \int(\ln|x|)\,\partial y + H_2(x) = y\ln|x| + H_2(x) \end{aligned}$$

In order to get these two expressions for $\phi(x,y)$ to agree, we must take $H_1(y) = 0$ and $H_2(x) = x^2$. Thus, $\phi(x,y) = x^2 + y\ln|x|$ and our implicit (general) solution is

$$x^2 + y\ln|x| = C$$

We now use the initial condition to fix the choice of the constant C.

$$y(2) = 1 \quad \Rightarrow \quad (2)^2 + (1)\ln|2| = C \quad \Rightarrow \quad C = 4 + \ln|2|$$

Thus, the implicit solution to the stated initial value problem is

$$x^2 + y\ln|x| = 4 + \ln|2|$$

6. (10 pts) Find an integrating factor for

$$2xy + \left(2x^2 + 2\right) \frac{dy}{dx} = 0$$

- We have

$$\begin{aligned} M(x,y) &= 2xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x \\ N(x,y) &= 2x^2 + 2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 4x \end{aligned}$$

The equation is not exact. However, it has both an integrating factor depending only on x and and integrating factor depending only on y. For,

$$F_1 \equiv \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - 4x}{2x^2 + 2} = -\frac{2x}{2x^2 + 2} = -\frac{x}{x^2 + 1} \quad \text{does not depend on } y$$

so

$$\begin{aligned} \mu(x) &= \exp\left(\int F_1(x)\,dx\right) = \exp\left(\int \frac{x}{x^2+1} dx\right) \\ &= \exp\left(\frac{1}{2}\int^{u=x^2+1} \frac{du}{u}\right) = \exp\left(\frac{1}{2}\ln\left(x^2+1\right)\right) \\ &= \sqrt{x^2+1} \end{aligned}$$

will be an integrating factor.

Also,

$$F_2 = F_1 \equiv \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - 2x}{2xy} = \frac{2x}{2xy} = \frac{1}{y} \quad \text{does not depend on } x$$

and so

$$\mu(y) = \exp\left(\int F_2(y)\,dy\right) = \exp\left(\int \frac{1}{y} dy\right) = \exp\left(\ln|y|\right) = y$$

will be an integrating factor.

7. (15 pts) Use a change of variables to solve

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$$

(Hint: try $z = y/x$.)

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$$z = \frac{y}{x} \quad \Rightarrow \quad y = zx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z$$

Substituting z for $\frac{y}{x}$ on the right hand side of the differential equation, and $x\frac{dz}{dx} + z$ for $\frac{dy}{dx}$ on the left hand side of our original differential equation we get

$$\begin{aligned} x\frac{dz}{dx} + z &= z + \frac{1}{z} \quad \Rightarrow \quad x\frac{dy}{dx} = \frac{1}{z} \quad \Rightarrow \quad z\frac{dz}{dx} = \frac{1}{x} \quad \Rightarrow \quad zdz = \frac{1}{x}dx \\ &\Rightarrow \quad \int zdz = \int \frac{1}{x}dx + C \\ &\Rightarrow \quad \frac{1}{2}z^2 = \ln|x| + C \\ &\Rightarrow \quad z = \pm\sqrt{2\ln|x| + 2C} \end{aligned}$$

We now replace z by its expression in terms of y and x

$$\frac{y}{x} = \pm\sqrt{2\ln|x| + 2C}$$

or

$$y = \pm x\sqrt{2\ln|x| + C'}$$