LECTURE 21

The Laplace Transform

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a "nice" (to be qualified latter) function of x. The **Laplace transform** $\mathcal{L}[f]$ of f is the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\, dx \quad . \tag{1}$$

We note that in the formula above, s is the variable upon which the Laplace transform $\mathcal{L}[f]$ depends.

Example 21.1. If

$$f(x) = ax$$

then

$$\begin{aligned} \mathcal{L}[f](s) &= \int_0^\infty axe^{-sx}\, dx \\ &= \lim_{N\to\infty} \left(-\tfrac{a}{s} xe^{-sx} - \tfrac{a}{s^2} e^{-sx}\right)\Big|_0^N \\ &= \tfrac{a}{s^2} \end{aligned}$$

Note that this result really only makes sense for $s > 0$; for $x \leq 0$ the integral does not converge.

Example 21.2. If

$$f(x) = \sin(ax)$$

then, integrating by twice by parts,

$$\begin{aligned} \mathcal{L}[f](s) &= \int_0^\infty \sin(ax) e^{-sx}\, dx \\ &= \lim_{N\to\infty} \left(e^{-sx} \tfrac{1}{a} \cos(ax)\right)\Big|_0^N + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\ &= \tfrac{1}{a} + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\ &= \tfrac{1}{a} + \lim_{N\to\infty} \tfrac{s}{a} \left(-\tfrac{1}{a} e^{-sx} \sin(ax)\right)\Big|_0^N - \tfrac{s^2}{a^2} \int_0^\infty e^{-sx} \sin(ax)\, dx \\ &= \tfrac{1}{a} + 0 - \tfrac{s^2}{a^2} L[f](s) \quad , \end{aligned}$$

we find

$$\mathcal{L}[f](s) = \frac{a}{1 + \frac{s^2}{a^2}} = \frac{a}{a^2 + s^2} \quad .$$

(If $s \leq 0$, the integral on the first line does not converge, so $\mathcal{L}[f](s)$ is only defined for $s > 0$.)

Example 21.3. If $f(x) = e^{bx}$, then

$$\begin{aligned} \mathcal{L}[f] &= \int_0^\infty e^{bt} e^{-st}\, dt \\ &= \int_0^\infty e^{(b-s)t} dt \\ &= \tfrac{1}{b-s} e^{(b-s)t}\Big|_0^\infty \\ &= \tfrac{1}{s-b} \quad (\text{if } s > b) \end{aligned}$$

(If $s \leq b$ then the integral does not converge.)

The following theorem explains under what conditions we can expect the Laplace transform of a function $f(x)$ to exist.

THEOREM 21.4. *Suppose that $f(x)$ is a piecewise continuous function for $0 \leq t \leq A$ and there exist constants K,a,M such that*

$$|f(t)| \leq Ke^{at} \quad , \quad \forall\, t > M > 0 \quad . \tag{2}$$

Then the Laplace transform $\mathcal{L}[f]$ defined by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\, dt$$

exists for all $s > a$.

The condition (2) is a rather moderate "growth" condition on the function $f(x)$; it says that for large enough t, $|f(t)|$ grows no faster than an exponential function of the form Ke^{at}. This condition is easily satisfied by any polynomial function of x.

THEOREM 21.5. ***Properties of the Laplace Transform***

(i) *Suppose $f_1(x)$ and $f_2(x)$ are two functions satisfying the hypotheses of Theorem 6.2. Then if $g(x) = c_1 f_1(x) + c_2 f_2(x)$, $\mathcal{L}[g]$ exists and*

$$\mathcal{L}[g](s) = c_1\mathcal{L}[f_1](s) + c_2\mathcal{L}[f_2](s) \quad .$$

(ii) *Suppose that f is continuous and that both f and its derivative f' satisfy the hypotheses of Theorem 6.2. Then $\mathcal{L}[f'](s)$ exists for $s > a$ and moreover*

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0) \quad .$$

(iii) *Suppose that f and its derivatives $f', \ldots, f^{(n-1)}$ are continuous and satisfy the hypotheses of Theorem 6.2. Then $\mathcal{L}[f^{(n)}](s)$ exists for $s > a$ and*

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \quad .$$

Proof of (i).

This follows from the linearity property integration:

$$\begin{aligned} \mathcal{L}[c_1 f_1 + c_2 f_2](s) &= \int_0^\infty \left(c_1 f_1(x) + c_2 f_2(x)\right) e^{-sx} dx \\ &= c_1 \int^x f_1(x)e^{-sx} dx \quad + \quad c_2 \int^x f_2(x)e^{-sx} dx \\ &= c_1\mathcal{L}[f_1](s) + c_2\mathcal{L}[f_2](s) \end{aligned}$$

Proof of (ii).

Integrating by parts one finds

$$\begin{aligned} \mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t) dt \\ &= e^{-st} f(t)\big|_0^\infty - \int_0^\infty \left(-se^{-st}\right) f(t) dt \\ &= 0 - f(0) + s\int_0^\infty e^{-st} f(t) dt \\ &= s\mathcal{L}[f] - f(0) \quad . \end{aligned}$$

Similarly, (iii) is proved by integrating by parts repeatedly.