LECTURE 19

Differential Equations with Polynomial Coefficients

In the last lecture we considered a number of examples of differential equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

and looked for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \quad . \tag{2}$$

Before considering one more example, let me first articulate the general procedure.

Step 1. Substitute (2) into (1). This will produce an equation of the form

$$\begin{aligned} 0 \;=\; & \textstyle\sum_{n=0}^{\infty} n(n-1)a_n P(x)(x-x_o)^{n-2} + \sum_{n=0}^{\infty} n a_n Q(x)(x-x_o)^{n-1} \\ & + \textstyle\sum_{n=0}^{\infty} a_n R(x)(x-1)^n \end{aligned} \tag{3}$$

Step 2. Unfortunately, depending on the nature of the polynomials, it may happen that none of three series in (3) is a power series in $(x - x_o)$. For example, if $P(x) = x^2$ and $x_o = 1$, then the first series is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^2 (x-1)^2 \tag{4}$$

which is not a power series (i.e., an expression of the form $\sum b_n(x-1)^n$ with each b_n a constant). To convert the series in (3) into to power series we must replace the polynomials $P(x)$, $Q(x)$, and $R(x)$ with their Taylor expansions about $x_o = 1$. If we set

$$\begin{aligned} p_n &= \tfrac{1}{n!}\tfrac{d^nP}{dx^n}(x_o) \\ q_n &= \tfrac{1}{n!}\tfrac{d^nQ}{dx^n}(x_o) \\ r_n &= \tfrac{1}{n!}\tfrac{d^nR}{dx^n}(x_o) \end{aligned}$$

we can write

$$\begin{aligned} P(x) &= \textstyle\sum_{i=0}^{\infty} p_n (x-x_o)^n \quad , \\ Q(x) &= \textstyle\sum_{i=0}^{\infty} q_n (x-x_o)^n \quad , \\ R(x) &= \textstyle\sum_{i=0}^{\infty} r_n (x-x_o)^n \quad . \end{aligned} \tag{5}$$

Actually, since polynomial of degree D can have at most D non-vanishing derivatives, each of the Taylor expansions (5) will terminate after a finite number of terms:

$$\begin{aligned} P(x) &= \textstyle\sum_{i=0}^{d_P} p_n (x-x_o)^n \quad , \\ Q(x) &= \textstyle\sum_{i=0}^{d_Q} q_n (x-x_o)^n \quad , \\ R(x) &= \textstyle\sum_{i=0}^{d_R} r_n (x-x_o)^n \quad . \end{aligned} \tag{6}$$

where d_P, d_Q, and d_R are the degrees of the polynomials $P(x)$, $Q(x)$, and $R(X)$. Inserting the expression (6) into (3) we get

$$\begin{aligned} 0 &= \textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_P} n(n-1)a_n p_n\,(x-x_o)^{n+i-2} \\ &\quad +\textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_P} n a_n q_n\,(x-x_o)^{n+i-1} \\ &\quad +\textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_R} a_n r_n\,(x-x_o)^{n+i} \end{aligned}$$

$$\begin{aligned} &= \textstyle\sum_{i=0}^{d_P}\sum_{n=0}^{\infty} n(n-1)a_n p_n\,(x-x_o)^{n+i-2} \\ &\quad +\textstyle\sum_{i=0}^{d_Q}\sum_{n=0}^{\infty} n a_n q_n\,(x-x_o)^{n+i-1} \\ &\quad +\textstyle\sum_{i=0}^{d_R}\sum_{n=0}^{\infty} a_n r_n\,(x-x_o)^{n+i} \end{aligned}$$

or

$$\begin{aligned} 0 &= \textstyle\sum_{n=0}^{\infty} n(n-1)p_0 a_n\,(x-x_o)^{n-2} + \cdots + \sum_{n=0}^{\infty} n(n-1)p_{d_P} a_n\,(x-x_o)^{n+d_P-2} \\ &\quad +\textstyle\sum_{n=0}^{\infty} n q_0 a_n\,(x-x_o)^{n-1} + \cdots + \sum_{n=0}^{\infty} n q_{d_Q} a_n\,(x-x_o)^{n+d_Q-1} \\ &\quad +\textstyle\sum_{n=0}^{\infty} r_0 a_n\,(x-x_o)^{n} + \cdots + \sum_{n=0}^{\infty} r_{d_R} a_n\,(x-x_o)^{n+d_R} \end{aligned} \tag{7}$$

Step 3. The next step is to collect all the terms consisting of like factors of $(x-x_o)^i$. To accomplish this we shift the summation index n in each series in (7) so that the k^{th} term in the new series has $(x-x_o)^k$ as a factor. One obtains

$$\begin{aligned} 0 &= \textstyle\sum_{k=-2}^{\infty}(k+2)(k+1)p_0 a_{k+2}\,(x-x_o)^k + \cdots \\ &\quad \cdots + \textstyle\sum_{n=-2+d_P}^{\infty}(k+2-d_P)(k+1-d_P)p_{d_P} a_{k+2-d_P}\,(x-x_o)^k \\ &\quad +\textstyle\sum_{k=0}^{\infty}(k+1)q_0 a_{k+1}\,(x-x_o)^k + \cdots + \sum_{k=-1+d_Q}^{\infty}(k+1-d_Q)\,q_{d_Q} a_{k+1-d_Q}\,(x-x_o)^k \\ &\quad +\textstyle\sum_{k=0}^{\infty} r_0 a_k\,(x-x_o)^k + \cdots + \sum_{k=d_R}^{\infty} r_{d_R} a_{k-d_R}\,(x-x_o)^k \end{aligned} \tag{8}$$

Here one must be a bit careful. Notice that the various series appearing in the above equation **do not** have the same initial value of k. Before consolidating the various series in (8) in a single series we must make sure they all start off at the same value of k. I will discuss this point momentarily with an example. But certainly for k large enough all the series in (8) will contribute terms proportional to $(x-x_o)^k$. One can then read off from (8) the general recursion relation

$$\begin{aligned} 0 &= (k+2)(k+1)p_o a_{k+2} + \cdots (k+2-d_P)(k+1-d_P)a_{k+2-d_P} \\ &\quad +(k+1)q_0 a_{k+1} + \cdots + (k+1-d_Q)q_{d_Q} a_{k+1-d_Q} \\ &\quad +r_0 a_k + \cdots + r_{d_R} a_{k-d_R} \end{aligned} \tag{9}$$

which is valid for $k > Max\{-2+d_P, -1+d_Q, d_R\}$. Actually, we can use this relation for all k so long as we consistently define

$$a_i = 0 \quad , \quad \text{if} \quad i < 0. \tag{10}$$

Step 4. Use the recursion relation (9) to express all the coefficients a_n in terms of a_0 and a_1 (you may also need to use the relations $0 = a_{-1} = a_{-2} = a_{-3}\cdots$ coming from (10)).

Example 19.1. Find a power series solution of

$$x^2 y'' + (x+1)y = 0 \tag{11}$$

about the point $x_o = 1$.

Plugging

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

into (11) yields

$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^2 (x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x+1)(x-1)^n \quad . \tag{12}$$

Now the Taylor expansions of $f(x) = x^2$ and $g(x) = x + 1$ about $x_o = 1$ are

$$\begin{aligned} x^2 &= 1 + 2(x-1) + (x-1)^2 \\ x+1 &= 2 + (x-1) \quad . \end{aligned} \tag{13}$$

Plugging the right hand sides of (13) into (12) yields

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n \left(1 + 2(x-1) + (x-1)^2\right)(x-1)^{n-2} \\ &\quad + \sum_{n=0}^{\infty} a_n \left(2 + (x-1)\right)(x-1)^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} 2n(n-1)a_n(x-1)^{n-1} \\ &\quad + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1} \end{aligned}$$

We now shift the summation indices in each series so that in the k^{th} term, $(x-1)$ appears to the k^{th} power. One gets

$$\begin{aligned} 0 &= 0 + 0 + \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k \quad + 0 + \sum_{k=0}^{\infty} 2(k+1)k(x-1)^k a_{k+1}(x-1)^k \\ &\quad + \sum_{k=0}^{\infty} k(k-1)a_k(x-1)^k + \sum_{k=0}^{\infty} 2a_k(x-1)^k + \sum_{k=1}^{\infty} a_{k-1}(x-1)^k \end{aligned} \tag{14}$$

Unfortunately, the last series begins with $k = 1$, instead of $k = 0$. This, however, is easy to remedy; we simply $a_{-1} = 0$, so that

$$\sum_{k=0}^{\infty} a_{k-1}(x-1)^k = 0(x-k)^{-1} + \sum_{k=1}^{\infty} a_{k-1}(x-1)^k = \sum_{k=1}^{\infty} a_{k-1}(x-1)^k \quad .$$

Thus, having arranged things so that all series start off at the same point $k = 0$ and we now consolidate the right hand side of (14) into a single series:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} + 2(k+1)ka_{k+1} + k(k-1)a_k + 2a_k + a_{k-1}\right)(x-1)^k \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} + 2k(k+1)a_{k+1} + (k^2 - k + 2)a_k + a_{k-1}\right)(x-1)^k \end{aligned}$$

The demand that the total coefficient of $(x-1)^k$ vanish then implies

$$a_{k+2} = \frac{-2k(k+1)a_{k+1} - (k^2 - k + 2)a_k - a_{k-1}}{(k+2)(k+1)} \quad .$$

Thus, given that $a_{-1} = 0$, we have

$$\begin{aligned} a_2 &= \tfrac{0 - 2a_0 - 0}{(2)(1)} = -a_0 \\ a_3 &= \tfrac{(-2)(2)a_2 - (2)a_1 - a_0}{(3)(2)} = \tfrac{-7a_0 - 2a_1}{6} \\ a_4 &= \tfrac{(-4)(3)a_3 - 4a_2 - a_1}{(4)(3)} = \tfrac{(14a_0 - 4a_1 + 4a_0 - a_1)}{12} = \tfrac{18a_0 - 5a_1}{12} \end{aligned}$$

Thus, to the order of $(x-1)^4$ the general solution of (11) is

$$\begin{aligned} y(x) &= a_0 + a_1(x-1) - a_0(x-1)^2 - \tfrac{7a_0 - 2a_1}{6}(x-1)^3 \\ &\quad + \tfrac{18a_0 - 5a_1}{12}(x-1)^4 + \cdots \\ &= a_0\left(1 - (x-1)^2 - \tfrac{7}{6}(x-1)^3 + \tfrac{3}{2}(x-1)^4 + \cdots\right) \\ &\quad + a_1\left((x-1) + \tfrac{1}{3}(x-1)^3 - \tfrac{5}{12}(x-1)^4 + \cdots\right) \end{aligned}$$