LECTURE 18

Solving Differential Equations Using Power Series

We are now going to employ power series to find solutions to differential equations of the form

(1) $$y'' + p(x)y' + q(x)y = 0$$

where the functions $p(x)$ and $q(x)$ are polynomials. Let us begin with a simple example.

EXAMPLE 18.1. Given that $y(x)$ satisfies

$$\begin{aligned} y'' + y' + x^2 y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

Write down the first four terms of the Taylor Expansion of $y(x)$ about $x = 0$.

The Taylor expansion of $y(x)$ about $x = 0$ is the power series

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \qquad (2) \\ &= y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{iv}(0)x^4 + \cdots \end{aligned}$$

From the initial conditions we know that

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

We can use the differential equation to determine the values of all the higher derivatives of $y(x)$ at $x = 0$. This is done as follows. Isolating the y'' term on the left hand side of the differential equation we obtain

(3) $$y''(x) = -y'(x) - x^2 y(x)$$

Evaluating (3) at $x = 0$ yields

(4) $$y''(0) = -(2) - (0)^2(1) = -2$$

To obtain $y'''(0)$ we can differentiate both sides of (3) and get

$$y'''(x) = -y''(x) = 2xy'$$

(5) $$y'''(x) = -y''(x) - 2xy(x) - x^2 y'(x)$$

We can now evaluate (5) at $x = 0$ to obtain

$$\begin{aligned} y'''(0) &= -y''(0) - 2(0)y(0) - (0)^2 y'(0) \\ &= -y''(0) \\ &= 2 \end{aligned}$$

where we have used (4) to evaluate $-y''(0)$.

Continuing in the same manner we can differentiate (5) to obtain

$$y^{iv}(x) = -y'''(x) - 2y(x) - 2xy(x) - 2xy'(x) - x^2 y''(x)$$

which upon evaluation at $x = 0$ yields

$$\begin{aligned} y^{iv}(0) &= -y'''(0) - 2y(0) - 2(0)y(0) - 2(0)y'(0) - 2(0)y'(0) - (0)^2 y''(0) \\ &= -y'''(0) - 2y(0) \\ &= -2 - 2 \\ &= -4 \end{aligned}$$

We now plug these results for $y(0), y'(0), y''(0), y'''(0)$, and $y^{iv}(0)$ into (2) and obtain

$$\begin{aligned} y(x) &= 1 + 2x + \frac{-2}{2}x^2 + \frac{2}{6}x^3 + \frac{-4}{24}x^4 + \cdots \\ &= 1 + 2x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots \end{aligned}$$

It should be clear that the method of this example can easily be extended to compute as many terms of the Taylor expansion of the solution of the differential equations as we want.

Example 18.2. In this example I will demonstrate an equivalent, but more systematic method of computing a power series expression for the solution of a differential equation.

Consider the follwing initial value problem:

$$\begin{aligned} y'' - y &= 0 \\ y(2) &= 1 \\ y'(2) &= -1 \end{aligned} \tag{6}$$

We shall look for a power series solution around $x_o = 2$; i.e., a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n \quad . \tag{7}$$

Our task is then is to determine the coefficients a_n so that this $y(x)$ indeed satisfies (6).

Now already the first two coefficients are determined by the initial conditions. To see this note

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x-2)^n \\ &= a_0 + a_1 (x-2) + a_2 (x-2)^2 + a_3 (x-2)^3 + \cdots \end{aligned}$$

and so

$$\begin{aligned} 1 = y(2) &= a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \cdots \\ &= a_0 \end{aligned}$$

Similarly,

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n (x-2)^{n-1} \\ &= 0 + a_1 + 2a_2 (x-2) + 3a_3 (x-2)^2 + \cdots \end{aligned}$$

and so

$$-1 = y'(2) = a_1 .$$

The remaining coefficients a_n can now be determined by the differential equation.

Differentiating term by term we have

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n (x-2)^n\right] \\ &= \frac{d}{dx}\left[a_0 + a_1 (x-2) + a_2 (x-2)^2 + a_3 (x-2)^3 + \cdots\right] \\ &= a_1 + 2a_2 (x-2) + 3a_3 (x-2)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \quad . \end{aligned}$$

Differentiating again one finds

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} \quad .$$

Thus, if (7) is to solve (6), we need

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} + \sum_{n=0}^{\infty} a_n (x-2)^n \quad .$$

To solve the equation above we first put the first power series in standard form:

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^n \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} (x-2)^m - \sum_{n=2}^{\infty} a_n (x-2)^n \\ &= \sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} - a_n)(x-2)^n \end{aligned}$$

This must vanish for all x, so the total coefficient of each power of $(x-2)^n$ must vanish separately:

$$(n+2)(n+1)a_{n+2} - a_n = 0 \quad , \quad \forall\, n.$$

We thus conclude

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n \quad . \quad \forall\, n.$$

A relationship of this type between the coefficients of a power series is called a **recursion relation.**

Here is how one applies the recursion relation to determine (almost) all of the coefficients a_n. To start, let's suppose a_0 and a_1 are given. Then

$$\begin{aligned} a_2 &= a_{0+2} = -\tfrac{1}{(0+2)(0+1)} a_0 = \tfrac{1}{2} a_0 \\ a_3 &= a_{1+2} = -\tfrac{1}{(1+2)(1+1)} a_1 = \tfrac{1}{3\cdot 2} a_1 \\ a_4 &= a_{2+2} = -\tfrac{1}{(2+2)(1+2)} a_2 = \tfrac{1}{4\cdot 3} a_1 = \tfrac{1}{4\cdot 3\cdot 2} a_0 \\ a_5 &= a_{3+2} = -\tfrac{1}{(3+2)(3+1)} a_3 = \tfrac{1}{5\cdot 4} a_3 = \tfrac{1}{5\cdot 4\cdot 3\cdot 2} a_1 \\ a_6 &= a_{4+2} = -\tfrac{1}{(4+2)(4+1)} a_4 = \tfrac{1}{6\cdot 5} a_4 = \tfrac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2} a_0 \end{aligned}$$

The following pattern emerges:

$$\begin{aligned} \text{if } n \text{ is even, i.e., } n = 2k, \text{ then } a_n &= \tfrac{1}{n!} a_o \\ \text{if } n \text{ is odd, i.e., } n = 2k+1, \text{ then } a_n &= \tfrac{1}{n!} a_1 \end{aligned}$$

But as we have already seen, the initial conditions imply

$$\begin{aligned} y(2) = 1 \quad &\Rightarrow \quad a_0 = 1 \\ y'(2) = 1 \quad &\Rightarrow \quad a_1 = -1 \end{aligned}$$

Thus,

$$a_n = \begin{cases} \frac{1}{n!} & \text{if } n \text{ is even,} \\ -\frac{1}{n!} & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$\begin{aligned} y(x) &= 1 - (x-2) + \tfrac{1}{2}(x-2)^2 - \tfrac{1}{3!}(x-2)^3 + \tfrac{1}{4!}(x-2)^4 + \cdots \\ &= \left(1 + \tfrac{1}{2!}(x-2)^2 + \tfrac{1}{4!}(x-2)^4 + \cdots\right) \\ &\quad - \left((x-2) + \tfrac{1}{3!}(x-2)^3 + \tfrac{1}{5!}(x2)^5 + \cdots\right) \\ &= \cosh(x-2) - \sinh(x-2) \end{aligned}$$

In the last step we have simply identified the power series within the parentheses as the Taylor expansions of $\cosh(x-2)$ and $\sinh(x-2)$.

This example generalizes as follows. To construct a power series solution around the point $x = x_o$, we procede as follows:

(1) Set $y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$.
(2) Plug the expression (1) for $y(x)$ into the differential equation;
(3) Manipulate the resulting equation to obtain an equation in which single power series expression (rather that a sum of several power series) is set equal to zero.
(4) Set the coefficient of each power of $(x - x_o)$ separately equal to zero. The resulting set of equations gives us the **recursion relation** for the coefficients a_n.

(5) Solve the recursion relations, one by one, starting with the equation for a_2 in terms of a_0 and a_1, until all the coefficients a_n are expressed in terms of a_0 and a_1.
(6) Write down the solution $y(x)$ using the results of preceding step.

A good question to ask at this point is: what is the special significance of a_0 and a_1? Here is the answer.

Recall that the equation

$$y'' + p(x)y' + q(x)y = 0$$

always has a unique solution satisfying given initial conditions

$$y(x_o) = y_o \quad , \quad y'(x_o) = y'_o$$

so long as $p(x)$, and $q(x)$ are well-behaved functions at x_o; in particular, if $p(x)$ and $q(x)$ are polynomials. Suppose we found a power series solution to this differential equation in the manner described above;

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \quad .$$

Then

$$\begin{aligned} y(x_o) &= \sum_{n=0}^{\infty} a_n (x_o - x_o)^n = a_0 + 0 + 0 + 0 + \cdots &= a_0 \quad , \\ y'(x_o) &= \sum_{n=0}^{\infty} n a_n (x_o - x_o)^{n-1} = 0 + a_1 + 0 + 0 + \cdots &= a_1 \quad . \end{aligned}$$

So the coefficients a_o and a_1 correspond exactly to the initial values of y and y' at x_o.

Example 18.3. Find a general solution of

$$y'' + xy' + y = 0$$

around $x_o = 1$.

We set

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

and plug into the differential equation:

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n \quad .$$

The hardest part about this problem will be to combine the three power series appearing on the right hand side of the equation above into a single power series. We have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (x-1+1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (x-1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=1} na_n(x-1)^{n-1} \\ &\quad + \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^n + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \\ &\quad + \sum_{n=0}^{\infty} a_n(x-x_o)^n \\ &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m + \sum_{n=0}^{\infty} na_n(x-1)^n \\ &\quad + \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k + \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0} na_n(x-1)^n \\ &\quad + \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1} + a_n\right)(x-1)^n \end{aligned}$$

We must therefore have

$$0 = (n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1} + a_n$$

or

$$a_{n+2} = -\frac{a_n + a_{n+1}}{(n+2)} \quad .$$

Thus,

$$\begin{aligned} a_2 &= -\tfrac{a_0+a_1}{2} \\ a_3 &= -\tfrac{a_1+a_2}{3} = -\tfrac{a_1}{3} + \tfrac{a_0+a_1}{6} = \tfrac{a_0-a_1}{6} \\ a_4 &= -\tfrac{a_2+a_3}{4} = \tfrac{a_0+a_1}{8} - \tfrac{a_0-a_1}{24} = \tfrac{a_0+2a_1}{12} \end{aligned}$$

So up to order four the general solution is

$$y(x) = a_0 + a_1(x-1) - \frac{a_0+a_1}{2}(x-1)^2 + \frac{a_0-a_1}{6}(x-1)^3 + \frac{a_0+2a_1}{12}(x-1)^4 + \cdots$$

or

$$\begin{aligned} y(x) &= a_0\left(1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 + \tfrac{1}{12}(x-1)^4 + \cdots\right) \\ &\quad + a_1\left(x - \tfrac{1}{2}(x-1)^2 - \tfrac{1}{6}(x-1)^3 + \tfrac{1}{6}(x-1)^4 + \cdots\right) \end{aligned}$$

EXAMPLE 18.4. Find a series solution of

$$y'' - xy = 0$$

about $x = 0$ satisfying the initial condition

$$y(0) = 1 \quad , \quad y'(0) = 2 \quad .$$

Plugging

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

into the differential equation yields

$$\begin{aligned} 0 &= \textstyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n \\ &= \textstyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \textstyle\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= \textstyle 2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= \textstyle 2a_2 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} - a_{n-1}\right)x^n \end{aligned}$$

Demanding that the total coefficient of each power of x must vanish leads to

$$\begin{aligned} a_2 &= 0 \\ a_{n+2} &= \tfrac{a_{n-1}}{(n+2)(n+1)} \end{aligned}$$

The latter equation can also be written as

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)} \quad .$$

Now the initial conditions imply

$$a_0 = 1 \quad , \quad a_1 = 2 \quad .$$

We also have $a_2 = 0$, and the recursion relation above yields

$$\begin{aligned} a_3 &= \tfrac{a_0}{(3)(2)} = \tfrac{1}{(2)(3)} = \tfrac{1}{6} \\ a_4 &= \tfrac{a_1}{(4)(3)} = \tfrac{2}{(4)(3)} = \tfrac{1}{6} \\ a_5 &= \tfrac{a_2}{(5)(4)} = \tfrac{0}{20} = 0 \\ a_6 &= \tfrac{a_3}{(6)(5)} = \tfrac{1}{(6)(5)}\tfrac{1}{6} = \tfrac{1}{180} \\ a_7 &= \tfrac{a_4}{(7)(6)} = \tfrac{1}{(7)(6)}\tfrac{1}{6} = \tfrac{1}{252} \end{aligned}$$

So up, to order seven,

$$y(x) = 1 + 2x + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{180}x^6 + \frac{1}{252}x^7 + \cdots$$

is a solution of the initial value problem.