LECTURE 17

Manipulating Power Series

Our technique for solving differential equations by power series will essentially be to substitute a generic power series expression

$$y(x)=\sum_{n=0}^{\infty}a_n\left(x-x_o\right)^n$$

into a differential equations and then use the consequences of this substitution to determine the coefficients a_n.

1. Differentiating Power Series

THEOREM 17.1. *If*

$$f(x)\equiv\sum_{n=0}^{\infty}a_n\left(x-x_o\right)^n$$

is a convergent power series with radius of convergence $R>0$ then it's derivative

$$\frac{df}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists for all $x\in(x_o-R,x_o+R)$ and is in fact equal to the power series

$$\sum_{n=0}^{\infty}na_n\left(x-x_o\right)^{n-1}$$

Moreover, the power series expression (5) for $f'(x)$ has the same radius of convergence as that of the original power series.

Thus, effectively, the derivative of a power series expression can be computed by simply differentiating term by term.

EXAMPLE 17.2. Consider the power series

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{n+1}x^n$$

This power series happens to converge for all $x\in(-1,1)$ and so for $x\in(-1,1)$, $f\left(x\right)$ is a differentiable function. Its derivative is

$$\frac{df}{dx}\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(nx^{n-1}\right)=\sum_{n=1}^{\infty}\frac{n}{n+1}x^{n-1}$$

with the power series on the right hand side converging for all $x\in(-1,1)$.

In summary, formally,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}a_n\left(x-x_o\right)^n\right)=\sum_{n=0}^{\infty}na_n\left(x-x_o\right)^{n-1}\tag{1}$$

2. Multiplying Power Series By Functions

When we substitute a power series expression into a differential equation we not only have to be able to differentiate power series, we sometimes have to multiply power series by functions of x.

EXAMPLE 17.3. If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, what is $xy'(x)$?

Well, according to our rule for differentiating power series

$$xy'(x) = x \sum_{n=0}^{\infty} n a_n x^{n-1} \quad .$$

The expression on the right hand side is not a power series expression though – it is x times a power series. Nevertheless, it is quite trivial to convert it to a power series expression. All we have to do is to distribute the factor of x over each term in the formal sum:

$$x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} x n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

Similarly,

$$x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}$$

and more generally

$$x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k} \quad .$$

However, things are not quite so simple when we have power series expanded about a point other than $x = 0$.

EXAMPLE 17.4. Find a power series expression for

$$x \sum_{n=0}^{\infty} a_n (x-1)^n \tag{2}$$

Although we can again bring the factor x through the summation sign, the resulting expression

$$\sum_{n=0}^{\infty} x a_n (x-1)^n$$

is not a power series expression (because the individual terms are not simply a constant times a power of $(x-1)$).

To make sense of the expression (2) we use the following trick:

$$x = x - 1 + 1 = 1 + (x-1)$$

and so

$$\begin{aligned} x \sum_{n=0}^{\infty} a_n (x-1)^n &= (1 + (x-1)) \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= (1) \sum_{n=0}^{\infty} a_n (x-1) \quad + \quad (x-1) \sum_{n=0}^{\infty} a_n (x-1)^{nn} \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n \quad + \quad \sum_{n=0}^{\infty} a_n (x-1)^{n+1} \end{aligned}$$

Thus, we can express (2) as a sum of two power series about $x = 1$. (I'll show below how to combine a sum of power series into a single power series expression).

EXAMPLE 17.5. Express $x^2 \sum_{n=0}^{\infty} a_n (x-1)^n$ as a sum of power series.

The essential trick we used in the preceding example was to rewrite x as $1 + (x-1)$. We now note the $1 + (x-1)$ is the Taylor expansion of $f(x) = x$ about $x = 1$: if we compute the Taylor expansion of $f(x) = x$ about $x = 1$ we get

$$\begin{aligned} f(x) &= f(1) + f'(x)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\ &= 1 + (1)(x-1) + (0)(x-1)^2 + (0)(x-1)^3 + \cdots \\ &= 1 + (x-1) \end{aligned}$$

Thinking about the preceding example in this way, the natural thing to try for $x^2 \sum_{n=0}^{\infty} a_n (x-1)^n$ is to replace x^2 by its Taylor expansion about $x = 1$. We have

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \quad \Rightarrow \quad f''(x) = 2 \quad \Rightarrow \quad f'''(x) = 0 \quad \Rightarrow \quad f^{(4)}(x) = 0 \quad \Rightarrow \quad \cdots$$

so

$$f(1) = 1 \quad , \quad f'(1) = 2 \quad , \quad f''(1) = 2 \quad \text{and} \quad f^{(i)}(2) = 0 \text{ for all } i > 2$$

Thus

$$\begin{aligned} x^2 &= 1 + 2(x-1) + \frac{2}{2!}(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + \cdots \\ &= 1 + 2(x-1) + (x-1)^2 \end{aligned}$$

Thus we can write

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} a_n (x-1)^n &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n \quad + \quad 2(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n \quad + \quad (x-1)^2 \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n \quad + \quad \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} \quad + \quad \sum_{n=0}^{\infty} a_n (x-1)^{n+2} \end{aligned}$$

More generally.

PROPOSITION 17.6 (Multiplying Power Series by Functions). *If $f(x)$ is a smooth function about x_0 with a finite Taylor expansion (e.g. if $f(x)$ is a polynomial function) and $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a convergent power series*

$$f(x) \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{m=0}^{M} \sum_{n=0}^{\infty} \frac{f^{(m)}(x_0)}{m!} a_n (x-x_0)^{n+m}$$

3. Adding Power Series

The basic rule for adding power series is as simple as could be:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (x-x_0)^n + \sum_{n=0}^{\infty} b_n (x-x_0) &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \\ &\quad + b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots \\ &= (a_0+b_0) + (a_1+b_1)(x-x_0) + (a_2+b_2)(x-x_0)^2 + \cdots \end{aligned}$$

or, consolidating the sum on the right via sigma notation,

PROPOSITION 17.7 (Adding Power Series). *If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n (x-x_0)^n$ are convergent power series then*

(3) $$\sum_{n=0}^{\infty} a_n (x-x_0)^n + \sum_{n=0}^{\infty} b_n (x-x_0)^n = \sum_{n=0}^{\infty} (a_n+b_n)(x-x_0)^n$$

Unfortunately, in practice we rarely get to use the simple rule (3) directly. Consider for example if we try to consolidate

$$x\sum_{n=0}^{\infty} = (1+(x-1))\sum_{n=0}^{n=0} a_n (x-1)^n = \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1}$$

into a single power series. The idea, of course, would be to collect the total coefficients of each distinct power of $(x-1)$, however, note that the $(x-1)^2$ occurs in $n=2$ term of $\sum_{n=0}^{\infty} a_n (x-1)^n$ but as the $n=1$ term of $\sum_{n=0}^{\infty} a_n (x-1)^{n+1}$. Our problem is that the n^{th} terms of the two power series involve different powers of $x-1$ and so we can't simply add the coefficient of the n^{th} term of one to the coefficient of the n^{th} term of the other as the formula (7) suggests.

To overcome this problem we will have to apply to separate operations.

3.1. Shifting Summation Indices.

PROPOSITION 17.8 (Shifting Summation Indices).

(4) $$\sum_{n=n_0}^{\infty} a_n (x-x_0)^{n+k} = \sum_{n=n_0+k} a_{n-k} (x-x_0)^n$$

Proof. Expanding both sides as formal sums we have

$$\begin{aligned}
\sum_{n=n_0}^{\infty} a_n (x-x_0)^{n+k} &= a_{n_0}(x-x_0)^{n_0+k} + a_{n_0+1}(x-x_0)^{n_0+k+1} + a_{n_0+2}(x-x_0)^{n_n+k+2} + \cdots \\
\sum_{n=n_0+k} a_{n-k} (x-x_0)^n &= a_{n_0-k+k}(x-x_0)^{n_0+k} + a_{n_0-k+1-k}(x-x_0)^{n_0+k+1} + a_{n_0-k+2+k}(x-x_0)^{n_0+k+2} + \cdots
\end{aligned}$$

Since

$$a_{n_0-k+k} = a_{n_0} \quad , \quad a_{n_0-k+1-k} = a_{n_0+1} \quad , \quad etc$$

its clear that these to formal sums are identical, and to the proposition follows.

Put another way. The formula (4) says that we can replace a power series with terms $a_n (x-x_0)^{n+k}$ by one with terms $a_{n-k}(x-x_0)^k$, we just need to remember that beside substituting $n-k$ for n in the individual terms, we have to change the starting value n_0 of n as well, and that the starting point n_0 gets shifted by the same amount but in the **opposite** direction: Thus,

$$\sum_{n=n_0}^{\infty} a_n (x-x_0)^{n+k} \quad \Rightarrow \quad \begin{array}{l}\text{substitute } n-k \text{ for } n \\ \text{substitute } n_0+k \text{ for } n_0\end{array} \quad \Rightarrow \quad \sum_{n=n_0+k} a_{n-k}(x-x_0)^n$$

EXAMPLE 17.9. Consolidate

$$\sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} n a_n (x-1)^{n-1}$$

into a single power series expression.

Using our shifting rule (4 we can rewrite the second power series as

$$\sum_{n=0}^{\infty} n a_n (x-1)^{n-1} \quad \Rightarrow \quad \begin{array}{c} n \to n+1 \\ n_0 \to n_0 - 1\end{array} \quad \Rightarrow \quad \sum_{n=0-1}^{\infty} (n-1) a_{n+1} (x-1)^{n+1-1} = \sum_{n=-1}^{\infty} (n+1) a_{n+1} (x-1)^n$$

Now note that the first term of the series on the right is actually equal to 0:

$$n = -1 \qquad \Rightarrow \qquad (n+1)\, a_{n+1}\, (x-1)^n = (-1+1)\, a_0\, (x-1)^{-1} = 0$$

Thus,

$$\sum_{n=-1}^{\infty} (n+1)\, a_{n+1}\, (x-1)^n = 0 + \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, (x-1)^n$$

and so

$$\sum_{n=0}^{\infty} a_n\, (x-1)^n + \sum_{n=0}^{\infty} n a_n\, (x-1)^{n-1} = \sum_{n=0}^{\infty} a_n\, (x-1)^n + \sum_{n=0} (n+1)\, a_{n+1}\, (x-1)^n$$

Now we can apply the simple rule (3) to add the power sereis on the right (because in both series the n^{th} term is some coefficient times $(x-1)^n$).

$$\sum_{n=0}^{\infty} a_n\, (x-1)^n + \sum_{n=0} (n+1)\, a_{n+1}\, (x-1) = \sum_{n=0}^{\infty} (a_n + (n+2)\, a_{n+1})\, (x-1)^n$$

Thus,

$$\sum_{n=0}^{\infty} a_n\, (x-1)^n + \sum_{n=0}^{\infty} n a_n\, (x-1)^{n-1} = \sum_{n=0}^{\infty} (a_n + (n+2)\, a_{n+1})\, (x-1)^n$$

3.2. Peeling Off Initial Terms. In the preceding example a subtle but crucial step occurred in the step where

$$\sum_{n=-1}^{\infty} (n+1)\, a_{n+1}\, (x-1)^n \quad \Rightarrow \quad 0 + \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, (x-1)^n$$

We needed to do this because the other series in the sum

$$\sum_{n=0}^{\infty} a_n\, (x-1)^n$$

began at $n = 0$ while, (at least at face value) the series

$$\sum_{n=-1}^{\infty} (n+1)\, a_{n+1}\, (x-1)$$

began at $n = -1$ and so had one extra term.

Occassionally, we will have to add series which start off at different values of n and for which the initial terms are not equal to 0. In such cases, we will have to keep track of the initial terms of one series "by hand" until the second series catches up.

3.3. Adding Power Series (reprise). As the preceding examples show, the hard part of adding two power series is manipulating the two series to a form where the simple rule (**) can be applied. Let me introduce some nomenclature to better delineate the steps in this process.

Definition 17.10. *We shall say that a power series is in* ***standard form*** *when the n^{th} term is of the form $a_n(x - x_o)^n$.*

We note that we can always add put a power series into standard form by performing a shift of summation indices. We note also that when we shift the summation index n up or down by k, the initial value n_0 in the summation moves, respectively, down or up by k.

The steps for adding power series expressions are

(i) Perform shifts-of-summation-indices so that all the power series to be added are in standard form.

(ii) Identify the power-series-in-standard-form that begins with the highest value of n. Call this highest initial value of n is n_0.
(iii) Peel-off by hand the beginning terms of the power-series-in-standard-forms 'til they catch up to the one that begins with $n = n_0$.
(iv) Collect together the total coefficient of each distinct power of $x - x_0$.

4. Examples

EXAMPLE 17.11. Put the following power series in standard form:

$$x \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} \quad . \tag{5}$$

The first thing we must do is make sure that all terms in the series involve only powers of $x-2$. We therefore replace the factor x by

$$x = (x-2) + 2 \quad .$$

Then (9) becomes

$$\begin{aligned} x \textstyle\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} &= \textstyle\sum_{n=0}^{\infty} n(n-1)\left((x-2)+2\right)a_n(x-2)^{n-2} \\ &= \textstyle\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-1} \\ &\quad + \textstyle\sum_{n=0}^{\infty} 2n(n-1)a_n(x-2)^{n-2} \end{aligned} \tag{6}$$

In order to put the two series on the right hand side in standard form, we shift the summation index to $k = n-1$ in the first series and shift the summation index to $k = n-2$ in the second index. We thus obtain

$$\begin{aligned} x \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} &= \sum_{k=-1}^{\infty} (k+1)(k)a_{k+1}(x-2)^k \\ &\quad + \sum_{k=-2}^{\infty} 2(k+2)(k+1)a_{k+2}(x-2)^k \end{aligned} \tag{7}$$

We'll next peel of the initial terms of the two series on the right so that both summations begin at $n = 0$:

$$\begin{aligned} \sum_{k=-1}^{\infty} (k+1)(k)a_{k+1}(x-2)^k &= (-1+1)(-1)(x-2)^{=1} + \sum_{k=0}^{\infty} k(k+1)a_{k+1}(x-2)^k \\ &= 0 + \sum_{k=0}^{\infty} k(k+1)a_{k+1}(x-2)^k \end{aligned}$$

$$\begin{aligned} \sum_{k=-2}^{\infty} 2(k+2)(k+1)a_{k+2}(x-2)^k &= 2(-2+2)(-2+1)a_{-2+2}(x-2)^{-2} + 2(-1+2)(-1+1)a_{-1+2}(x-2)^{-1} + \sum_{k=0}^{\infty} 2(k - \\ &= 0 + 0 + \sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}(x-2)^k \end{aligned}$$

$$\begin{aligned} x \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} &= (-1+1)(-1)(x-2)^{=1} + \sum_{k=0}^{\infty} k(k+1)a_{k+1}(x-2)^k \\ &\quad + 2(-2+2)(-2+1)a_{-2+2}(x-2)^{-2} + 2(-1+2)(-1+1)a_{-1+2}(x-2)^{-1} + \sum_{k=0}^{\infty} 2(k+2)(\end{aligned}$$

Making these substitutions in the right hand side of (7) we obtain

$$x \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{k=0}^{\infty} k(k+1)a_{k+1}(x-2)^k \quad + \quad \sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}(x-2)^k \tag{8}$$

Both power series on the right hnad side of (8) are in standard form, and so we can use the formula (3) to combine them into a single power series:

$$x\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{k=0}^{\infty}\left[k(k+1)a_{k+1} \; + \; 2(k+2)(k+1)a_{k+2}\right](x-2)^k \tag{9}$$

The expression on the right hand side of (9) is now in the standard form

$$\sum_{k=0}^{\infty} A_k(x-x_o)^k$$

with

$$\begin{aligned} x_o &= 2 \quad , \\ A_k &= k(k+1)a_{k+1} + 2(k+1)(k+2)a_{k+2} \end{aligned}$$

and so we're done.

Example 17.12. Transform

$$x^2\sum_{n=0} a_n\,(x-1)^n \tag{10}$$

into single power series expression in standard form.

First we replace x^2 by its Taylor expansion about $x = 1$;

$$x^2 = 1 + 2\,(x-1) + (x-1)^2$$

Then we have

$$\begin{aligned} x^2\sum_{n=0}^{\infty} a_n\,(x-1)^n &= \left(1 + 2\,(x-1) + (x-1)^2\right)\sum_{n=0}^{\infty} a_n\,(x-1)^n \\ &= (1)\sum_{n=0}^{\infty} a_n\,(x-1)^n + 2\,(x-1)\sum_{n=0}^{\infty} a_n\,(x-1)^n + (x-1)^2\sum_{n=0}^{\infty} a_n\,(x-1)^n \\ &= \sum_{n=0}^{\infty} a_n\,(x-1)^n + \sum_{n=0}^{\infty} 2a_n\,(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n\,(x-1)^{n+2} \end{aligned}$$

We've now rewritten the original expression (10) as a sum of three power series. The next step will be to shift summation indices on the last two series so that each series in the sum is in standard form:

$$\sum_{n=0}^{\infty} 2a_n\,(x-1)^{n+1} \quad \xrightarrow{\quad n \to n-1 \quad} \quad \sum_{n=1}^{\infty} 2a_{n-1}\,(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n\,(x-1)^{n+2} \quad \xrightarrow{\quad n \to n-2 \quad} \quad \sum_{n=2}^{\infty} a_{n-2}\,(x-1)^n$$

and so

$$x^2\sum_{n=0}^{\infty} a_n\,(x-1) = \sum_{n=0}^{\infty} a_n\,(x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1}\,(x-1)^n + \sum_{n=2}^{\infty} a_{n-2}\,(x-1)^n$$

Before we can combine the power series summations on the right hand side we need to make sure that the summations all start off at the same value of n. We'll "peel off" the first two terms of the first series (so that it's summation has "caught up" with the summation in the last series, and the peel off the first term of the second series so that it too has caught up with the last series. Thus

$$
\begin{aligned}
x^2\sum_{n=0}^{\infty}a_n\left(x-1\right) &= \sum_{n=0}^{\infty}a_n\left(x-1\right)^n \\
&\quad +\sum_{n=1}^{\infty}2a_{n-1}\left(x-1\right)^n \\
&\quad +\sum_{n=2}^{\infty}a_{n-2}\left(x-1\right)^n \\
&= a_0\left(x-1\right)^0+a_1\left(x-1\right)^1+\sum_{n=2}^{\infty}a_n\left(x-1\right)^n \\
&\quad +2a_0\left(x-1\right)^1+\sum_{n=2}^{\infty}2a_{n-1}\left(x-1\right)^n \\
&\quad +\sum_{n=0}^{\infty}a_{n-2}\left(x-1\right)^n \\
&= a_0+\left(a_1+2a_0\right)\left(x-1\right)+\sum_{n=2}^{\infty}\left(a_n+2a_{n-1}+a_{n-2}\right)\left(x-1\right)^n
\end{aligned}
$$

The expression on the far right **is** a power series expression; it is just a formal each term of which is some constant coefficient times a distinct power of $x-1$. Thus,

$$
x^2\sum_{n=0}^{\infty}a_n\left(x-1\right)=a_0+\left(a_1+2a_0\right)\left(x-1\right)+\sum_{n=2}^{\infty}\left(a_n+2a_{n-1}+a_{n-2}\right)\left(x-1\right)^n
$$

is the answer to the problem posed.