LECTURE 15

# Nonhomogeneous Second Order Linear Differential Equations

We now consider differential equations of the form

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

where $g(x) \neq 0$.

Obvious things that we'd like to know are

- how to construct solutions; and
- how to know if we have all the solutions.

To this end, it certainly would be nice to have something like the Superposition Principle at our disposal. However, for non-homogeneous linear differential equations the Superposition Principle can not be applied. To see this, suppose $Y_1(x)$ and $Y_2(x)$ are solutions of (1). If the Superposition Principle were valid then $Y(x) = c_1Y_1(x) + c_2Y_2(x)$ would also be a solution. But for this $Y(x)$

$$\begin{aligned}
Y'' + p(x)Y' + q(x)Y &= c_1Y_1'' + c_2Y_2'' + p(x)\left(c_1Y_1' + c_2Y_2'\right) + q(x)\left(c_1Y_1 + c_2Y_2\right) \\
&= c_1\left(Y_1'' + p(x)Y_1' + q\left(x\right)Y_1\right) + c_2\left(Y_2'' + p(x)Y_2' + q(x)Y_2\right) \\
&= c_1g(x) + c_2g(x) \\
&= \left(c_1 + c_2\right)g(x) \\
&\neq g(x)
\end{aligned}$$

Thus, if $y_1(x)$ and $y_2(x)$ satisfy (1) then a linear combinattion of $y_1$ and $y_2$ doesn's satisfy the same equation. So we can't make more solutions out of two independent solutions.

However, the calculation carried above, nevertheless, leads us to a way of constructing the general solution to (1). Let $Y_1(x)$ and $Y_2(x)$ be any two solutions of (1) and consider the function $\Delta Y(x)$ defined as the difference of $Y_1(x)$ and $Y_2(x)$:

$$\Delta Y(x) = Y_1(x) - Y_2(x)\,.$$

Applying the calculation above with $c_1 = 1$ and $c_2 = -1$ we see that $Y(x)$ obeys

$$Y'' + p(x)Y' + q(x)Y = (1-1)g(x) = 0$$

Thus, the difference of any two solutions of (1) must be a solution of the corresponding homogeneous equation.

Now let's assume that we've solved the correpsponding homogeneous equation

$$(\Delta y)'' + p(x)\left(\Delta Y\right)' + q(x)\Delta Y = 0. \tag{2}$$

From the general theory of homogeneous second order linear equations, we know that every solution can be expressed as a linear combination of two linearly independent solutions. Suppose then that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of (2) so that any solution of (2) can be expressed in the form $c_1y_1(x) + c_2y_2(x)$. This implies in particular that the function $\Delta Y(x) = Y_1(x) - Y_2(x)$ must be expressible in the form $c_1y_1(x) + c_2y_2(x)$ (since as we have seen $y(x)$ satisfies (2)). Thus, we have

$$Y_1(x) - Y_2(x) = y(x) = c_1y_1(x) + c_2y_2(x) \tag{3}$$

(Note that the functions $Y_1(x)$ and $Y_2(x)$ on the left hand side are solutions of the non-homogeneous equation (1) and the functions $y_1(x)$ and $y_2(x)$ on the right hand side are two linearly independent solutions of the corresponding homogeneous equation (2)). Now, in this calculation both $Y_1(x)$ and $Y_2(x)$ are arbitrary solutions of the non-homogeneous equation (1). Let's now interprete $Y_2(x)$ as some fixed solution $Y_p(x)$ of (1) and interprete $Y_1(x)$ as representing any other solution $Y(x)$ of (1). Then equation (3) becomes

$$Y(x) - Y_p(x) = c_1y_1(x) + c_2y_2(x)$$

or

$$Y(x) = Y_p(x) + c_1y_1(x) + c_2y_2(x).$$

Thus, every solution of the non-homogeneous equation (1) can be expressed in terms of a single solution $Y_p(x)$ and a linear combination of two linearly independent solutions of the corresponding homogeneous problem (2).

The following two theorems summarize these results and provide the foundation by which we construct solutions of such non-homogeneous second order ODEs.

Theorem 15.1. *If $Y_1$ and $Y_2$ are two solutions of the nonhomogeneous equation*

$$y'' + p(x)y' + q(x)y = g(x) \quad , \tag{4}$$

*then their difference*

$$Y(x) = Y_1(x) - Y_2(x)$$

*is a solution of the corresponding homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{5}$$

*If, in addition, $y_1$ and $y_2$ are a fundamental set of solutions to (5), then*

$$Y_1(x) - Y_2(x) = c_1y_1(x) + c_2y_2(x) \quad .$$

Theorem 15.2. *Given one solution $y_p(x)$ of the nonhomogeneous differential equation*

$$y'' + p(x)y' + q(x)y = g(x)$$

*then any other solution of this equation can be expressed as*

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x)$$

*where $y_1$ and $y_2$ are two linearly independent solutions of the corresponding homogeneous differential equation*

$$y'' + p(x)y' + q(x)y = 0 \quad .$$

Thus, to determine the general solution of a non-homogeneous linear equation (4), we can procede in three steps.

(1) Determine the general solution $c_1y_1(x) + c_2y_2(x)$ of the corresponding homogeneous problem.
(2) Find a particular solution $y_p(x)$ of the nonhomogeneous differential equation (5).
(3) Construct the general solution of (4) by setting

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) \quad .$$

Example 15.3. Given that one solution of

$$y'' + 3y' + 2y = e^{-x} \tag{6}$$

is $y(x) = xe^{-x}$, write down the general solution.

To construct the general solution we just apply the preceding theorem. We can immediately identify the function $y_p(x)$ in the theorem statement with our given solution $xe^{-x}$. To write down the general solution we also need two linearly independent solutions of

$$y'' + 3y' + 2 = 0\,. \tag{7}$$

Luckily, this is an equation we can solve. The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 3\lambda + 2 = 0$$

which factors as

$$(\lambda + 2)(\lambda + 1) = 0$$

Thus, we have two real roots $\lambda = -2, -1$ and hence two linearly solutions of (7)

$$\begin{aligned} y_1(x) &= e^{-2x} \\ y_2(x) &= e^{-x} \end{aligned}$$

We now have all the ingredients we need to write down the general solution of (6):

$$\begin{aligned} y(x) &= y_p(x) + c_1 y_1(x) + c_2 y_2(x) \\ &= xe^{-x} + c_1 e^{-2x} + c_2 e^{-x} \end{aligned}$$

## 1. Variation of Parameters

Consider again the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{8}$$

Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous problem corresponding to (8); i.e., $y_1$ and $y_2$ satisfy

$$y'' + p(x)y' + q(x)y = 0 \tag{9}$$

and

$$W[y_1, y_2] \neq 0 \quad . \tag{10}$$

We seek to determine two functions $u_1(x)$ and $u_2(x)$ such that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{11}$$

is a solution of (8). To determine the two functions $u_1$ and $u_2$ uniquely we need to impose two (independent) conditions. First, we shall require (11) to be a solution of (8); and second, we shall require

$$u_1' y_1 + u_2' y_2 = 0 \quad . \tag{12}$$

(This latter condition is imposed not only because we need a second equation, but also to make the calculation a lot easier.)

Differentiating (11) yields

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \tag{13}$$

which because of (13) becomes

$$y_p' = u_1 y_1' + u_2 y_2' \quad . \tag{14}$$

Differentiating again yields

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \quad . \tag{15}$$

We now plug (110, (13), and (15) into the original differential equation (8).

$$\begin{aligned} g(x) &= (u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'') + p(x)\,(u_1 y_1' + u_2 y_2') + q(x)\,(u_1 y_1 + u_2 y_2) \\ &= u_1' y_1' + u_2' y_2' + u_1\,(y_1'' + p(x) y_1' + q(x) y_1) + u_2\,(y_2'' + p(x) y_2' + q(x) y_2) \end{aligned}$$

The last two terms vanish since $y_1$ and $y_2$ are solutions of $y'' + py' + qy = 0$. We thus have

$$u_1' y_1 + u_2' y_2 = 0 \tag{16}$$

$$u_1' y_1' + u_2' y_2' = g \tag{17}$$

We now can now solve this pair of equations for $u_1$ and $u_2$. Rather than explicitly carry out the algebraic solution of equations (16) and (17), we'll use the following general fact:

FACT 15.4. *Let*

$$\begin{aligned} Ax + By &= e \\ Cx + Dy &= f \end{aligned}$$

*be a pair of independent linear equations in two unknowns $x$ and $y$. Then the solution of this system is given by*

$$\begin{aligned} x &= \frac{eD - Bf}{AD - BC} \\ y &= \frac{Af - eC}{AD - BC} \end{aligned}$$

Thus, in the situation at hand, regarding (16) and (17) as a pair of linear equations for $u_1'$ and $u_2'$, we have

$$\begin{aligned} u_1' &= \frac{-y_2 g}{y_1 y_2' - y_1' y_2} = \frac{-y_2 g}{W[y_1, y_2]} \\ u_2' &= \frac{y_1 g}{y_1 y_2' - y_1' y_2} = \frac{y_1 g}{W[y_1, y_2]} \quad . \end{aligned}$$

(Note that division by $W(y_1, y_2)$ causes no problems since $y_1$ and $y_2$ were chosen such that $W(y_1, y_2) \neq 0$.) Hence

$$\begin{aligned} u_1(x) &= \int^x \frac{-y_2(t) g(t)}{W[y_1, y_2](t)} dt \\ u_2(x) &= \int^x \frac{y_1(t) g(t)}{W[y_1, y_2](t)} dx' \end{aligned}$$

and so

$$\boxed{y_p(x) = -y_1(x) \int^x \frac{y_2(t) g(t)}{W[y_1, y_2](t)} dt + y_2(x) \int^x \frac{y_1(t) g(t)}{W[y_1, y_2](t)} dt} \tag{18}$$

is a particular solution of (8).

EXAMPLE 15.5. Find the general solution of

$$y'' - y' - 2y = 2e^{-x} \tag{19}$$

using the method of Variation of Parameters.

Well, the corresponding homogeneous problem is

$$y'' - y' - 2y = 0 \quad . \tag{20}$$

This is a second order linear equation with constant coefficients whose characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \quad .$$

The characteristic equation has two distinct real roots

$$\lambda = -1, 2$$

and so the functions

$$\begin{aligned} y_1(x) &= e^{-x} \\ y_2(x) &= e^{2x} \end{aligned}$$

form a fundamental set of solutions to (20).

To find a particular solution to (19) we employ the Variation of Parameters formula (18). Now

$$g(x) = 2e^{-x}$$

and

$$W[y_1, y_2](x) = \left(e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right) = 3e^x \quad ,$$

so

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(t) g(t)}{W[y_1, y_2](t)} dt \quad + \quad y_2(x) \int^x \frac{y_1(t) g(t)}{W[y_1, y_2](t)} dt \\ &= -e^{-x} \int^x \frac{e^{2t}\left(2e^{-t}\right)}{3e^t} dt \quad + \quad e^{2x} \int^x \frac{e^{-t}\left(2e^{-t}\right)}{3e^t} dt \\ &= -e^{-x} \int^x \frac{2}{3} dt \quad + \quad e^{2x} \int^x \frac{2}{3} e^{-3t} dt \\ &= -\frac{2}{3} x e^{-x} - \frac{2}{9} e^{-x} \end{aligned}$$

The general solution of (19) is thus

$$
\begin{aligned}
y(x) &= y_p(x) + c_1 y_1(x) + c_2(x) \\
&= -\tfrac{2}{3} x e^{-x} + \left(c_1 - \tfrac{2}{9}\right) e^{-x} + c_2 e^{2x} \\
&= -\tfrac{2}{3} x e^{-x} + C_1 e^{-x} + C_2 e^{2x}
\end{aligned}
$$

where we have absorbed the $-\frac{2}{9}$ in the second line into the arbitrary parameter $C_1$.